ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ *such that* $\min\{a+b,b+c,c+a\}\ge 2$, then :

$$2\left(\sqrt[2]{1,25^{a+b+c}}-1\right)\ge\sqrt[3]{\frac{(abc)^2}{(ab+1)(bc+1)(ca+1)}}$$

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$$(a+b)+(b+c)+(c+a)\ge 3\min\{a+b,b+c,c+a\}\ge 6\Rightarrow\sum_{cyc}a\ge 3$$

$$\Rightarrow\frac{\sum_{cyc}a}{3}\ge 1\Rightarrow 2\left(\sqrt[2]{1,25^{a+b+c}}-1\right)=2\left(\left(\frac{5}{4}\right)^{\frac{\sum_{cyc}a}{3}}-1\right)\overset{\text{Bernoulli}}{\ge}$$

$$2\left(1+\frac{1}{4}\cdot\frac{\sum_{cyc}a}{3}-1\right)=\frac{\sum_{cyc}a}{6}\overset{A-G}{\ge}\frac{\sqrt[3]{abc}}{2}\overset{?}{\ge}\sqrt[3]{\frac{(abc)^2}{(ab+1)(bc+1)(ca+1)}}$$

$$\Leftrightarrow(ab+1)(bc+1)(ca+1)\overset{?}{\ge}8abc\Leftrightarrow a^2b^2c^2+abc\sum_{cyc}a+\sum_{cyc}ab+1\underset{(*)}{\overset{?}{\ge}}8abc$$

$$\text{Now, LHS of }(*)\overset{A-G}{\ge}(abc)^2+3(abc)^{\frac{4}{3}}+3(abc)^{\frac{2}{3}}+1\overset{?}{\ge}8abc$$

$$\Leftrightarrow t^6+3t^4+3t^2+1\overset{?}{\ge}8t^3\ \left(t=\sqrt[3]{abc}\right)\Leftrightarrow t^6+3t^4-8t^3+3t^2+1\overset{?}{\ge}0$$

$$\Leftrightarrow(t-1)^2\left(t^4+2t^3+6t^2+2t+1\right)\overset{?}{\ge}0\to\text{true}\because t>0\Rightarrow(*)\text{ is true}$$

$$\therefore 2\left(\sqrt[2]{1,25^{a+b+c}}-1\right)\ge\sqrt[3]{\frac{(abc)^2}{(ab+1)(bc+1)(ca+1)}}\ \forall\, a,b,c>0$$

$$\left|\min\{a+b,b+c,c+a\}\ge 2,''=''\text{ iff } a=b=c=1\ (\text{QED})\right.$$