ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then :

$$4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3 \geq 7$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x$
$=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \rightarrow (1) \Rightarrow a=s-x,$
$b=s-y, c=s-z \therefore abc=r^2s \rightarrow (2)$

$$\therefore 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3 \overset{\text{via (1) and (2)}}{=} 4\sum_{cyc}\frac{s-x}{x}$$

$$+\frac{r^2s}{s^3}\left(\sum_{cyc}\frac{x}{y}\right)^3 \overset{A-G}{\geq} \frac{4s(s^2+4Rr+r^2)}{4Rrs}-12+\frac{9r^2}{s^2}\left(\frac{1}{xyz}\sum_{cyc}\frac{x^2z^2}{z}\right)$$

$$\overset{\text{Bergstrom}}{\geq} \frac{s^2+4Rr+r^2}{Rr}-12+\frac{9r^2}{s^2}\left(\frac{1}{4Rrs}\cdot\frac{(s^2+4Rr+r^2)^2}{2s}\right) \overset{?}{\geq} 7$$

$$\Leftrightarrow \frac{s^2+4Rr+r^2}{Rr}+\frac{9r(s^2+4Rr+r^2)^2}{8Rs^4} \overset{?}{\geq} 19$$

$$\Leftrightarrow \boxed{8s^6-(120Rr+17r^2)s^4+r^2s^2(72R^2+18r^2)+9r^4(4R+r)^2 \overset{(\bullet)}{\geq} 0} \text{ and}$$

$\because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(\bullet)$, it suffices to prove :

$$\text{LHS of } (\bullet) \geq 8(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (264Rr-103r^2)s^4-r^2s^2(6144R^2-3912Rr+582r^2)$$

$$+r^3(32768R^3-30576R^2r+9672Rr^2-991r^3) \overset{(\bullet\bullet)}{\geq} 0 \text{ and}$$

$\because (264Rr-103r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(\bullet\bullet)$,
it suffices to prove : LHS of $(\bullet\bullet) \geq (264Rr-103r^2)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (288R^2-253Rr+56r^2)s^2 \overset{(\bullet\bullet\bullet)}{\geq} r(4352R^3-4754R^2r+1676Rr^2-198r^3)$$

Now, $(288R^2-253Rr+56r^2)s^2 \overset{\text{Gerretsen}}{\geq} (288R^2-253Rr+56r^2)(16Rr-5r^2)$

$$\overset{?}{\geq} r(4352R^3-4754R^2r+1676Rr^2-198r^3)$$

$$\Leftrightarrow 256t^3-734t^2+485t-82 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(145t^2+111t(t-2)+41) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet)$$

$$\text{is true} \therefore 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3 \geq 7\ \forall\, a,b,c>0,$$

$$''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Schur's inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{(a^3+b^3+c^3+3abc)+a^2(b+c)+b^2(c+a)+c^2(a+b)}{(a+b)(b+c)(c+a)}$$

$$\geq \frac{2[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{(a+b)(b+c)(c+a)}$$

$$=2-\frac{4abc}{(a+b)(b+c)(c+a)}.$$

By AM − GM inequality, we have

$$(a+b)(b+c)(c+a)\geq \frac{8(a+b+c)(ab+bc+ca)}{9}\geq \frac{8\sqrt{3abc(a+b+c)^3}}{9}.$$

Using these inequalities and by AM − GM inequality, we obtain

$$4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq 8-2\sqrt{\frac{27abc}{(a+b+c)^3}}\geq 8-\left(1+\frac{27abc}{(a+b+c)^3}\right)$$

$$=7-\frac{27abc}{(a+b+c)^3}.$$

Now by AM − GM inequality, we have

$$abc\left(\frac{\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}}{a+b+c}\right)^3\geq abc\left(\frac{3}{a+b+c}\right)^3=\frac{27abc}{(a+b+c)^3}.$$

Adding the last two inequalities yields the desired result.

Equality holds iff $a=b=c$.