ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\ge 0$ such that $ab+bc+ca=1$. Prove that:

$$\frac{1}{\sqrt{a+bc}}+\frac{1}{\sqrt{b+ac}}+\frac{1}{\sqrt{c+ab}}\ge\frac{6}{\sqrt{2(a+b+c)+abc}}.$$

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Let us denote $p=a+b+c\ge\sqrt{3(ab+bc+ca)}=\sqrt{3}$ and

$r=abc\ge\max\left\{0,\frac{p(4-p^2)}{4}\right\}$ (Schur's Inequality).

Using B.C.S Inequality we have:

$$\frac{1}{\sqrt{a+bc}}+\frac{1}{\sqrt{b+ac}}+\frac{1}{\sqrt{c+ab}}\ge\frac{9}{\sqrt{a+bc}+\sqrt{b+ac}+\sqrt{c+ab}}\ge\frac{9}{\sqrt{3(a+b+c+1)}}$$

$$=\frac{3\sqrt{3}}{\sqrt{a+b+c+1}}=\frac{3\sqrt{3}}{\sqrt{p+1}};$$

We need to prove that:

$$\frac{3\sqrt{3}}{\sqrt{p+1}}\ge\frac{6}{\sqrt{2p+r}}\quad(*)$$

$$\Leftrightarrow 3(2p+r)\ge 4(p+1)\Leftrightarrow 2p+3r\ge 4;$$

- If $p\ge 2$ then $r\ge\max\left\{0,\frac{p(4-p^2)}{4}\right\}=0$ .We have:
  $2p+3r\ge 4+0=4$ (true) $\Rightarrow (*)$ true.
- If $\sqrt{3}\le p\le 2$ then $r\ge\max\left\{0,\frac{p(4-p^2)}{4}\right\}=\frac{p(4-p^2)}{4}$. We have:
  $2p+3r\ge 2p+3.\frac{p(4-p^2)}{4}=5p-\frac{3p^3}{4}$. We just prove that:

$$5p-\frac{3p^3}{4}\ge 4\Leftrightarrow 3p^3-20p+16\le 0\Leftrightarrow(p-2)(3p^2+6p-8)\le 0;$$

$\Leftrightarrow (p-2)[(3p+9)(p-1)+1]\le 0$ (true since $\sqrt{3}\le p\le 2$)

$\Rightarrow (*)$ true. Proved. Equality $\Leftrightarrow\begin{cases}a+b+c=2\\ab+bc+ca=1\\abc=0\end{cases}$

$\Leftrightarrow a=b=1, c=0$ or $b=c=1=0$ or $a=c=1, b=0$.