ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$ then:

$$\sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \le 2$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$LHS = \sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \overset{X+Y+Z=1}{\cong}$$

$$\sqrt{x(x+y+z) + yz} + \sqrt{y(x+y+z) + xz} + \sqrt{z(x+y+z) + xy}$$

$$= \sqrt{x^2 + xy + xz + yz} + \sqrt{y^2 + xy + xz + yz} + \sqrt{z^2 + xz + xy + yz}$$

$$= \sqrt{x(x+z) + y(x+z)} + \sqrt{x(y+z) + y(y+z)} + \sqrt{x(y+z) + z(y+z)}$$

$$= \sqrt{(x+z)(x+y)} + \sqrt{(y+z)(x+y)} + \sqrt{(y+z)(x+z)}$$

$$\overset{AM-GM}{\cong} \frac{2x + y + z}{2} + \frac{2y + x + z}{2} + \frac{2z + x + y}{2} = 2(x + y + z) = 2.1 = 2 = RHS$$

$$Hence: \sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy} \le 2, =iff: x = y = z = \frac{1}{3}$$

Solution 2 by Sakirin Ly-Cambodgia

$$Method\ 1: using\ AM - GM$$

$$we - get: \sqrt{\frac{4}{9}}S = \sqrt{\frac{4}{9}(x + yz)} + \sqrt{\frac{4}{9}(y + zx)} + \sqrt{\frac{4}{9}(z + xy)}$$

$$\frac{2}{3}S \le \frac{\frac{4}{9} + (x + yz)}{2} + \frac{\frac{4}{9} + (y + zx)}{2} + \frac{\frac{4}{9} + (z + xy)}{2}$$

$$\frac{2}{3}S \le \frac{\frac{4}{3} + (x + y + z) + (xy + yz + zx)}{2} \le \frac{\frac{4}{3} + 1 + \frac{(x+y+z)^2}{3}}{2} = \frac{\frac{5}{3} + 1}{2} = \frac{4}{3}$$

$$\to S \le \frac{4}{3}.\frac{3}{2} = 2 \to S \le 2\ \ (Equalities - when: x = y = z = \frac{1}{3})$$

$$Method2: using\ C - B - S$$

$$S^2 = (\sqrt{x + yz} + \sqrt{y + xz} + \sqrt{z + xy})^2 \le (1 + 1 + 1)(x + y + z + xy + yz + zx)$$

$$\le 3\left(1 + \frac{(x + y + z)^2}{3}\right) \to S^2 \le 3.\frac{4}{3} = 4 => S \le 2$$

$$(Equalities - when: x = y = z = \frac{1}{3})$$

Solution 3 by Samed Ahmedov-Azerbaijan

$$x=1-(y+z)=>\sqrt{x+yz}=\sqrt{1-(y+z)+yz}$$

$$=\sqrt{(1-y)(1-z)}\overset{AM-GM}{\overbrace{\leq}}\frac{2-(y+z)}{2}$$

$$also\ \sqrt{y+xz}=\sqrt{(1-x)(1-z)}\leq\frac{2-(x+z)}{2}$$

$$\sqrt{z+xy}=\sqrt{(1-x)(1-y)}\leq\frac{2-(x+y)}{2}$$

$$\sqrt{x+yz}+\sqrt{y+xz}+\sqrt{z+xy}\leq\frac{6-2(x+y+z)}{2}=\frac{4}{2}=2$$

$$Equalities-when:x=y=z=\frac{1}{3}$$