ROMANIAN MATHEMATICAL MAGAZINE

Let $\{x, y, z\}$ be positive real numbers such that : $x^2 + y^2 + z^2 = 3.$

$$\text{Prove that : } \frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z} \le 1$$

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$$4-x, 4-y, 4-z \ge 4-\sqrt{3} > 0 \Rightarrow (4-x)(4-y)(4-z) > 0$$

$$\therefore \frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z} \le 1$$

$$\Leftrightarrow 48 - 8\sum_{\text{cyc}} x + \sum_{\text{cyc}} xy \le 64 - 16\sum_{\text{cyc}} x + 4\sum_{\text{cyc}} xy - xyz$$

$$\Leftrightarrow 3\sum_{\text{cyc}} xy + 16 \ge 8\sum_{\text{cyc}} x + xyz \overset{x^2+y^2+z^2=3}{\Leftrightarrow} 3\sum_{\text{cyc}} xy + \frac{16}{3}.\left(\sum_{\text{cyc}} x^2\right) \ge$$

$$\frac{8}{\sqrt{3}}\left(\sum_{\text{cyc}} x\right).\sqrt{\sum_{\text{cyc}} x^2} + \frac{xyz.\sqrt{3}}{\sqrt{\sum_{\text{cyc}} x^2}} = \frac{8\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} x^2\right) + 3xyz}{\sqrt{3\sum_{\text{cyc}} x^2}}$$

$$\Leftrightarrow \frac{1}{9}\left(9\sum_{\text{cyc}} xy + 16\sum_{\text{cyc}} x^2\right)^2 \overset{(*)}{\ge} \frac{1}{3\sum_{\text{cyc}} x^2}.\left(8\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} x^2\right) + 3xyz\right)^2$$

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{\text{cyc}} x = \sum_{\text{cyc}} a = 2s \Rightarrow \sum_{\text{cyc}} x \overset{(1)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$

$\Rightarrow xyz \overset{(2)}{=} r^2 s$ and via such substitutions, $\sum_{\text{cyc}} xy = \sum_{\text{cyc}} (s-a)(s-b) = 4Rr + r^2$

$$\Rightarrow \sum_{\text{cyc}} xy \overset{(3)}{=} 4Rr + r^2 \Rightarrow \sum_{\text{cyc}} x^2 = \left(\sum_{\text{cyc}} x\right)^2 - 2\sum_{\text{cyc}} xy = s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{\text{cyc}} x^2 \overset{(4)}{=} s^2 - 8Rr - 2r^2 \therefore (1),(2),(3),(4) \Rightarrow$$

$$(*) \Leftrightarrow (s^2 - 8Rr - 2r^2)\left(16(s^2 - 8Rr - 2r^2) + 9(4Rr + r^2)\right)^2$$

$$\ge 3(8s(s^2 - 8Rr - 2r^2) + 3r^2 s)^2$$

$$\Leftrightarrow 32s^6 - (960Rr + 312r^2)s^4 + r^2 s^2(9864R^2 + 5508Rr + 747r^2)$$

$$-r^3(33856R^3+25392R^2r+6348Rr^2+529r^3)\boxed{\overset{(**)}{\geq}}\,0$$

$$and \because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \geq 32(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (576R-792r)s^4 - r(14712R^2-20868Rr+1653r^2)s^2 +$$

$$r^2(97216R^3-148272R^2r+32052Rr^2-4529r^3)\boxed{\overset{(***)}{\geq}}\,0$$

$$and \because (576R-792r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (576R-792r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (3720R^2-10236Rr+6267r^2)s^2 \boxed{\overset{(****)}{\geq}}$$

$$r(50240R^3-146640R^2r+109068Rr^2-15271r^3)$$

$$\text{Now, } 3720R^2-10236Rr+6267r^2 = (R-2r)(3720R-2796r)+675r^2 \overset{\text{Euler}}{\geq}$$

$$675r^2 > 0 \therefore \text{LHS of } (****) \overset{\text{Gerretsen}}{\geq} (3720R^2-10236Rr+6267r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(50240R^3-146640R^2r+109068Rr^2-15271r^3)$$

$$\Leftrightarrow 1160t^3-4467t^2+5298t-2008 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(1160t+173)+1350\big) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z} \leq 1$$

$$\forall\, x,y,z>0 \,\big|\, x^2+y^2+z^2=3, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$