ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x^2 + y^2 + z^2 = 3$ then:

$$\frac{1}{4-x} + \frac{1}{4-y} + \frac{1}{4-z} \le 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Hikmat Mammadov-Azerbaijan

$$x^2 + y^2 + z^2 = 3$$

$$\rightarrow We\ set: \begin{cases} x = \sqrt{3}\sin\theta\cos\alpha \\ y = \sqrt{3}\sin\theta\cos\alpha \\ z = \sqrt{3}\cos\theta \end{cases} \rightarrow \theta, \quad \alpha \in R \rightarrow Then: \ S = \frac{1}{4-x} + \frac{1}{4-y} + \frac{1}{4-z} =$$

$$= \frac{1}{4-\sqrt{3}\sin\theta\cos\alpha} + \frac{1}{4-\sqrt{3}\sin\theta\cos\alpha} + \frac{1}{4-\sqrt{3}\cos\theta}$$

$$Note: \ \theta\ has\ nothing\ to\ do\ with\ \alpha.$$

$$Hence\ we\ adjust\ for\ Local\ Maximum\ value\ as\ treating\ \theta, \alpha\ constant.$$

$$\rightarrow S(\alpha) = \frac{A}{B-\cos\alpha} + \frac{A}{B-\sin\alpha} + C \rightarrow S'(\alpha) = A.\left[\frac{\sin\alpha}{(B-\cos\alpha)^2} - \frac{\cos\alpha}{(B-\sin\alpha)^2}\right]$$

$$Let: \ S'(\alpha) = 0, clearly\ we\ have\ \sin\alpha = \cos\alpha, we\ take\ \alpha = 45^\circ = \frac{\pi}{4}$$

$$\rightarrow \ Hence: S(\alpha) \le S\left(\frac{\pi}{4}\right) = \frac{1}{4-\frac{\sqrt{6}}{2}\sin\theta} + \frac{1}{4-\frac{\sqrt{6}}{2}\sin\theta} + \frac{1}{4-\sqrt{3}\cos\theta} =$$

$$= \frac{4}{8-\sqrt{3}\sin\theta} + \frac{1}{4-\sqrt{3}\cos\theta}$$

$$Use\ the\ same\ trick\ to\ \theta \rightarrow \ S'(\theta) = \frac{\sqrt{3}\sin\theta}{(4-\sqrt{3}\cos\theta)^2} - \frac{4\sqrt{6}\cos\theta}{(8-\sqrt{6}\sin\theta)^2}$$

$$Let: \ S'(\theta) = 0 \ : \ \sin\theta = \frac{2}{\sqrt{6}}\ and\ \cos\theta = \frac{1}{\sqrt{3}}$$

$$\rightarrow \ Hence: S = S(\alpha) \le S(\theta) \le \frac{1}{4-\frac{\sqrt{6}}{2}.\frac{2}{\sqrt{6}}} + 2.\frac{1}{4-\sqrt{3}.\frac{1}{\sqrt{3}}} = 1$$

$$Therefore: \ \frac{1}{4-x} + \frac{1}{4-y} + \frac{1}{4-z} \le 1 \rightarrow proved.$$

Solution 2 by Pham Duc Nam-Vietnam

$$p = x + y + z, \qquad q = xy + yz + zx, \qquad r = xyz$$

$$x^2 + y^2 + z^2 = 3 \rightarrow p^2 - 2q = 3 \rightarrow q = \frac{p^2 - 3}{2}$$

$$\textit{And we also have}: (x + y + z)^2 \le 3(x^2 + y^2 + z^2) = 3\,.\,3 = 9\ \ p \le 3$$

$$\frac{1}{4 - x} + \frac{1}{4 - y} + \frac{1}{4 - z} \le 1 \rightarrow$$

$$(4 - y)(4 - z) + (4 - x)(4 - z) + (4 - x)(4 - y) \le (4 - x)(4 - y)(4 - z) \rightarrow$$

$$8(x + y + z) - 3(xy + yz + zx) + xyz - 16 \le 0 \rightarrow 8p - 3q + r - 16 \le 0$$

$$\textit{By pqr transformation we have}: r \le \frac{1}{27}p^3 \rightarrow$$

$$8p - 3q + r - 16 \le 8p - 3q + \frac{1}{27}p^3 - 16$$

$$\textit{Now, replace } q = \frac{p^2 - 3}{2} \rightarrow 8p - 3q + r - 16 \le 8p - 3.\frac{p^2 - 3}{2} + \frac{1}{27}p^3 - 16 =$$

$$\frac{1}{54}(p - 3)^2(2p - 69) \le 0$$

$$\textit{This is true since } (p - 3)^2 \ge 0 \textit{ and } p \le 3 \rightarrow 2p - 69 < 0$$

$$\textit{Equality holds iff } x = y = z = 1.$$

Solution 3 by Le Thu-Vietnam

$$\textit{By tangent line method,} \qquad \textit{we claim that}:$$

$$f(x) = \frac{18}{x^2 + 5} \overset{?}{\le} f(1) + f'(1)(x - 1) = 4 - x \rightarrow \frac{1}{4 - x} \overset{?}{\le} \frac{18}{x^2 + 5} <-> (2 - x)(x - 1)^2 \ge 0$$

$$\textit{Which is true } \forall x \in (0, \sqrt{3} < 2).$$

$$\therefore x, y \textit{ and } z > 0: \sum \frac{1}{4 - x} \le \sum\left(\frac{x^2 + 5}{8}\right) \overset{\sum x^2 = 3}{=} 1$$

$$\textit{Equality holds iff } x = y = z = 1.$$