ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0, xyz=1$ then:

$$\sqrt[3]{\frac{x}{y+7}}+\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{x+7}}\ge\frac{3}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Tapas Das-India

$$\left(\frac{y+7}{8}\right)^{\frac{1}{3}}=\left(1+\frac{y-1}{8}\right)^{\frac{1}{3}}\overset{Bernoulli}{\le}1+\frac{y-1}{24}=\frac{y+23}{24}$$

$$\left(\frac{8}{y+7}\right)^{\frac{1}{3}}\ge\frac{24}{y+23}\ or\ \frac{1}{\sqrt[3]{y+7}}\ge\frac{12}{y+23}\ (1)$$

$$\forall x,y,z>0\ \left(\sum x^2\right)^2\ge3\sum x^3y\,(Vasc\ inequality)$$

$$\sqrt[3]{\frac{x}{y+7}}+\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{x+7}}\overset{(1)}{\ge}\sum\frac{12\sqrt[3]{x}}{y+23}\ (2)$$

$$let\ x^{\frac{1}{3}}=\frac{a}{b},y^{\frac{1}{3}}=\frac{c}{a},z^{\frac{1}{3}}=\frac{b}{c}\ then\ from\ (2):$$

$$\sqrt[3]{\frac{x}{y+7}}+\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{x+7}}\overset{(1)}{\ge}\sum\frac{12\sqrt[3]{x}}{y+23}=12\sum\frac{\left(\frac{a}{b}\right)}{\frac{c^3}{a^3}+23}=$$

$$=12\sum\frac{a^4}{b(c^3+23a^3)}\overset{Bergstrom}{\ge}12\frac{(a^2+b^2+c^2)^2}{(\sum bc^3)+23(\sum a^3b)}\ge$$

$$\overset{(2)}{\ge}12\frac{(a^2+b^2+c^2)^2}{\frac{(\sum a^2)^2}{3}+\frac{23(\sum a^2)^2}{3}}=\frac{3}{2}$$

$$(Equality\ holds\ for\ a=b=c=1)$$