ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$, $x^2+y^2+z^2=3$ then:

$$\frac{yz}{4-x^2}+\frac{zx}{4-y^2}+\frac{xy}{4-z^2}\le 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Tapas Das-India

$$4-x^2\overset{x^2+y^2+z^2=3}{=}=1+x^2+y^2+z^2-x^2=1+y^2+z^2\overset{AM-GM}{\ge}1+2yz\ (1)$$

$$\frac{yz}{4-x^2}+\frac{zx}{4-y^2}+\frac{xy}{4-z^2}=\sum\frac{yz}{4-x^2}\overset{(1)}{\le}\sum\frac{yz}{1+2yz}=$$

$$=\frac{1}{2}\sum\left(1-\frac{1}{1+2yz}\right)=\frac{3}{2}-\frac{1}{2}\sum\frac{1^2}{1+2yz}\overset{Bergstrom}{\le}$$

$$\le\frac{3}{2}-\frac{1}{2}\frac{(1+1+1)^2}{3+2(xy+yz+zx)}=\frac{3}{2}-\frac{1}{2}\frac{9}{3+2\sum xy}\le\frac{3}{2}-\frac{1}{2}\frac{9}{3+2\sum x^2}=$$

$$=\frac{3}{2}-\frac{1}{2}.\frac{9}{3+2.3}(as\ \ x^2+y^2+z^2=3\ )=\frac{3}{2}-\frac{1}{2}=1$$

Equality holds for $x=y=z=1$

Solution 2 by Lamiye Quliyeva-Azerbaijan

$$\frac{yz}{4-x^2}+\frac{xz}{4-y^2}+\frac{xy}{4-z^2}\le\frac{yz}{4-(3-y^2-z^2)}+\frac{xz}{4-(3-x^2-z^2)}+$$

$$+\frac{xy}{4-(3-x^2-y^2)}\le\frac{yz}{1+y^2+z^2}+\frac{xz}{1+x^2+z^2}+\frac{xy}{1+x^2+y^2}$$

$$\left.\begin{array}{l}\dfrac{yz}{1+y^2+z^2}\le\dfrac{x^2}{3}\\ \dfrac{xz}{1+x^2+z^2}\le\dfrac{y^2}{3}\\ \dfrac{xy}{1+x^2+y^2}\le\dfrac{z^2}{3}\end{array}\right\}\Rightarrow\frac{yz}{1+y^2+z^2}+\frac{xz}{1+x^2+z^2}+\frac{xy}{1+x^2+y^2}\le\frac{x^2}{3}+\frac{y^2}{3}+\frac{z^2}{3}$$

$$\frac{x^2}{3}+\frac{y^2}{3}+\frac{z^2}{3}\le\frac{x^2+y^2+z^2}{3}\le 1$$

$$\frac{yz}{4-x^2}+\frac{xz}{4-y^2}+\frac{xy}{4-z^2}\le 1$$

$$Equality\ holds\ for\ x=y=z=1$$

Solution 3 by Ertan Yildirim-Turkiye

$$\frac{yz}{4-x^2}+\frac{xz}{4-y^2}+\frac{xy}{4-z^2}\le 1$$

$$\frac{2yz}{4-x^2}+\frac{2xz}{4-y^2}+\frac{2xy}{4-z^2}\le 2$$

$$\frac{2yz}{4-x^2}+\frac{2xz}{4-y^2}+\frac{2xy}{4-z^2}\overbrace{\le}^{A-G}\frac{y^2+z^2}{4-x^2}+\frac{x^2+z^2}{4-y^2}+\frac{x^2+y^2}{4-z^2}=$$

$$\frac{3-x^2}{4-x^2}+\frac{3-y^2}{4-y^2}+\frac{3-z^2}{4-z^2}=1-\frac{1}{4-x^2}+1-\frac{1}{4-y^2}+1-\frac{1}{4-z^2}=$$

$$3-\left(\frac{1}{4-x^2}+\frac{1}{4-y^2}+\frac{1}{4-z^2}\right)\overbrace{\le}^{Bergstrom}3-\frac{(1+1+1)^2}{12-(x^2+y^2+z^2)}=$$

$$3-\frac{9}{12-3}=3-\frac{9}{9}=2$$

$$Equality\ \ holds\ \ for\ \ x=y=z=1$$