ROMANIAN MATHEMATICAL MAGAZINE

If $\{x, y, z\}$ be non − negative real nmbers such that : $x + y + z = 3$,

then prove that : $\frac{12-8x^2}{1+yz} + \frac{12-8y^2}{1+xz} + \frac{12-8z^2}{1+xy} \le 6$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 1 variable equals to zero and WLOG we may assume

$x = 0$ $(y + z = 3$ with $y, z > 0)$ and then : $\frac{12-8x^2}{1+yz} + \frac{12-8y^2}{1+xz} + \frac{12-8z^2}{1+xy} \le 6$

$$\Leftrightarrow \frac{6}{1+yz} + 9 \le 4\left((y+z)^2 - 2yz\right) \overset{y+z=3}{\Leftrightarrow} \frac{6}{1+t} + 9 \le 4(9-2t) \ (t = yz)$$

$$\Leftrightarrow 8t^2 - 19t - 21 \ge 0 \Leftrightarrow t \ge \frac{19-\sqrt{1033}}{16} \approx -0.82127 \text{ and } t \le \frac{19-\sqrt{1033}}{16}$$

$$\approx 3.19627 \to \text{true} \because t \in \left(0, \frac{9}{4}\right] \left(\because 3 = y + z \overset{A-G}{\ge} 2.\sqrt{yz} \Rightarrow \sqrt{t} \le \frac{3}{2} \Rightarrow t \le \frac{9}{4}\right)$$

$$\therefore \frac{12-8x^2}{1+yz} + \frac{12-8y^2}{1+xz} + \frac{12-8z^2}{1+xy} < 6$$

$\boxed{\text{Case 2}}$ Exactly 2 variables equal to zero and WLOG we may assume $y = z = 0$

$$(x = 3) \text{ and then : } \frac{12-8x^2}{1+yz} + \frac{12-8y^2}{1+xz} + \frac{12-8z^2}{1+xy} \le 6 \Leftrightarrow \frac{12-8.9}{1} + 12 + 12$$

$$= -36 < 6 \therefore \frac{12-8x^2}{1+yz} + \frac{12-8y^2}{1+xz} + \frac{12-8z^2}{1+xy} < 6$$

$\boxed{\text{Case 3}}$ $x, y, z > 0$ and then : $\frac{12-8x^2}{1+yz} + \frac{12-8y^2}{1+xz} + \frac{12-8z^2}{1+xy} \le 6$

$$\Leftrightarrow 6\sum_{cyc} \frac{1}{1+yz} \le 4\sum_{cyc} \frac{x^2}{1+yz} + 3 \to \text{(i)}$$

$$\sum_{cyc} \frac{1}{1+yz} = \frac{\sum_{cyc}\left((1+zx)(1+xy)\right)}{(1+xy)(1+yz)(1+zx)} = \frac{xyz\sum_{cyc} x + 2\sum_{cyc} xy + 3}{x^2y^2z^2 + xyz\sum_{cyc} x + \sum_{cyc} xy + 1}$$

$$\overset{x+y+z=3}{=} \frac{3xyz + 2\sum_{cyc} xy + 3}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1} \Rightarrow 6\sum_{cyc} \frac{1}{1+yz} - 3$$

$$= \frac{18xyz + 12\sum_{cyc} xy + 18}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1} - 3$$

$$\Rightarrow 6\sum_{cyc} \frac{1}{1+yz} - 3 \overset{(*)}{=} \frac{9xyz + 9\sum_{cyc} xy + 15 - 3x^2y^2z^2}{x^2y^2z^2 + 3xyz + \sum_{cyc} xy + 1} \text{ and}$$

$$4\sum_{cyc} \frac{x^2}{1+yz} = \frac{4\sum_{cyc}\left(x^2(1+zx)(1+xy)\right)}{(1+xy)(1+yz)(1+zx)}$$

$$=\frac{4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-4xyz\sum_{cyc}x+4\sum_{cyc}x^2}{x^2y^2z^2+xyz\sum_{cyc}x+\sum_{cyc}xy+1}$$

$$\overset{x+y+z=3}{=}\frac{4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-12xyz+4\sum_{cyc}x^2}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}$$

$$\therefore 4\sum_{cyc}\frac{x^2}{1+yz}\overset{(**)}{=}\frac{4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-12xyz+4\sum_{cyc}x^2}{x^2y^2z^2+3xyz+\sum_{cyc}xy+1}$$

$$\therefore (*),(**)\Rightarrow (\mathrm{i})\Leftrightarrow 4xyz\sum_{cyc}x^3+4\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-12xyz+4\sum_{cyc}x^2$$

$$\geq 9xyz+9\sum_{cyc}xy+15-3x^2y^2z^2\overset{x+y+z=3}{\Leftrightarrow}$$

$$4xyz\sum_{cyc}x^3+\frac{4}{9}\cdot\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-\frac{12xyz}{27}\cdot\left(\sum_{cyc}x\right)^3$$

$$+\frac{4}{81}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)^4\geq\frac{9xyz}{27}\cdot\left(\sum_{cyc}x\right)^3$$

$$+\frac{9}{81}\cdot\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^4+\frac{15}{729}\cdot\left(\sum_{cyc}x\right)^6-3x^2y^2z^2$$

$$\Leftrightarrow \boxed{\begin{gathered}972xyz\sum_{cyc}x^3+108\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-189xyz\left(\sum_{cyc}x\right)^3+\\ 12\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)^4\overset{(\bullet)}{\geq}27\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^4+5\left(\sum_{cyc}x\right)^6-729x^2y^2z^2\end{gathered}}$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(1)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$

$$\Rightarrow xyz\overset{(2)}{=}r^2s$$

Via such substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2$

$$\Rightarrow\sum_{cyc}xy\overset{(3)}{=}4Rr+r^2\Rightarrow\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2\left(4Rr+r^2\right)\Rightarrow\sum_{cyc}x^2\overset{(4)}{=}s^2-8Rr-2r^2\text{ and}$$

$$\sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3(x+y)(y+z)(z+x) \overset{\text{via (1)}}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} x^3 \overset{(5)}{=} s(s^2 - 12Rr) \therefore \text{via } (1),(2),(3),(4) \text{ and } (5), (\bullet) \Leftrightarrow$$

$$972r^2s^2(s^2 - 12Rr) + 108(s^2 - 8Rr - 2r^2)(4Rr + r^2)s^2 - 189r^2s^4$$
$$+12(s^2 - 8Rr - 2r^2)s^4 \geq 27(4Rr + r^2)s^4 + 5s^6 - 729r^4s^2$$

$$\Leftrightarrow \boxed{7s^4 + rs^2(228R + 840r) \overset{(\bullet\bullet)}{\geq} r(3456R^2 + 13392Rr - 513r^2)}$$

$$\text{Now, LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq} \Big(7(16Rr - 5r^2) + r(228R + 840r)\Big)s^2 \overset{\text{Gerretsen}}{\geq}$$

$$(340Rr + 805r^2)(16Rr - 5r^2) \overset{?}{\geq} r(3456R^2 + 13392Rr - 513r^2)$$

$$\Leftrightarrow 496R^2 - 553Rr - 878r^2 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(496R + 439r) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6 \text{ and}$$

$$\text{combining } all \text{ cases}, \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$$

$$\forall \text{ non-negative } \{x, y, z\} \text{ such that} : x + y + z = 3, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Since: } \frac{12-8x^2}{1+yz} = 12 - \frac{4(2x^2+3yz)}{1+yz} \text{ (and analogs)},$$

then the desired inequality is equivalent to

$$\frac{2x^2 + 3yz}{1 + yz} + \frac{2y^2 + 3zx}{1 + zx} + \frac{2z^2 + 3xy}{1 + xy} \geq \frac{15}{2}.$$

Let $p := x + y + z = 3$, $q := xy + yz + zx$, $r := xyz$. **By AM − GM inequality, we have**

$$\frac{2x^2 + 3yz}{1 + yz} + \frac{(2x^2 + 3yz)(1 + yz)}{4} \geq 2x^2 + 3yz \Rightarrow \frac{2x^2 + 3yz}{1 + yz} \geq \frac{6x^2 + 9yz - 2x^2yz - 3y^2z^2}{4}.$$

$$\Rightarrow \sum_{cyc} \frac{2x^2 + 3yz}{1 + yz} \geq \sum_{cyc} \frac{6x^2 + 9yz - 2x^2yz - 3y^2z^2}{4} = \frac{6(p^2 - 2q) + 9q - 2pr - 3(q^2 - 2pr)}{4}$$

$$\overset{p=3}{=} \frac{54 + 12r - 3q - 3q^2}{4} \overset{?}{\geq} \frac{15}{2} \Leftrightarrow 8 + 4r \geq q + q^2.$$

▪**If** $q \leq 2$, **we have** $q + q^2 \leq 2 + 2^2 < 8 \leq 8 + 4r$.

▪**If** $2 \leq q$. **Since** $3q \leq p^2 = 9$, **then** $q \leq 3$, **and by Schur's inequality, we have**

$$8+4r\geq 8+4.\frac{p(4q-p^2)}{9}=8+\frac{4(4q-9)}{3}=q+q^2+\frac{(3-q)(3q-4)}{3}\geq q+q^2.$$

So the proof is complete. Equality holds iff $x=y=z=1$.