Let $\{x, y, z\}$ be positive real nmbers such that : $xyz = 1$. Prove that :

$$\frac{7+x}{x^2+2x+1}+\frac{7+y}{y^2+2y+1}+\frac{7+z}{z^2+2z+1}\ge 6$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{7+x}{x^2+2x+1}+\frac{7+y}{y^2+2y+1}+\frac{7+z}{z^2+2z+1}=\sum_{cyc}\frac{x+1+6}{(x+1)^2}$$

$$=\sum_{cyc}\frac{1}{x+1}+6\left(\left(\sum_{cyc}\frac{1}{x+1}\right)^2-2\sum_{cyc}\frac{1}{(y+1)(z+1)}\right)$$

$$=\frac{3+2m+n}{2+m+n}+6\left(\left(\frac{3+2m+n}{2+m+n}\right)^2-\frac{2(m+3)}{2+m+n}\right)$$

$$\left(m=\sum_{cyc}x, n=\sum_{cyc}xy \text{ and } \because xyz=1\right)\overset{?}{\ge} 6$$

$$\Leftrightarrow 8m^2+3mn+n^2-5m-19n-36\underbrace{\overset{?}{\ge}}_{(*)}0$$

Case 1 $m \ge n$ and then : LHS of $(*) = 8m^2-5m-57+3mn+n^2-19n+21$

$$\overset{m\ge n}{\ge}(8m+19)(m-3)+4n^2-19n+21$$

$$=(8m+19)(m-3)+(4n-7)(n-3)\ge 0$$

$$\because m=\sum_{cyc}x\overset{A-G}{\ge}3.\sqrt[3]{xyz}\overset{xyz=1}{=}3 \text{ and } n=\sum_{cyc}xy\overset{A-G}{\ge}3.\sqrt[3]{x^2y^2z^2}\overset{xyz=1}{=}3$$

$$\Rightarrow m,n\ge 3\Rightarrow (*) \text{ is true}$$

Case 2 $n \ge m$ and then : LHS of $(*) = 8m^2-5m-36+3mn+n^2-19n\overset{n\ge m}{\ge}$

$$8m^2-5m-36+3m^2+m^2-19n\overset{m^2\ge 3n}{\ge}8m^2-5m-36+12n-19n$$

$$\overset{m^2\ge 3n}{\ge}8m^2-5m-36-\frac{7m^2}{3}=\frac{17m^2-15m-108}{3}=\frac{(17m+36)(m-3)}{3}\ge 0$$

$$\because m=\sum_{cyc}x\overset{A-G}{\ge}3.\sqrt[3]{xyz}\overset{xyz=1}{=}3\Rightarrow m\ge 3\Rightarrow (*) \text{ is true } \therefore \text{ combining both cases,}$$

$$(*) \text{ is true } \forall\, x,y,z>0 \text{ such that : } xyz=1$$

$$\therefore \frac{7+x}{x^2+2x+1}+\frac{7+y}{y^2+2y+1}+\frac{7+z}{z^2+2z+1}\ge 6\ \forall\, x,y,z>0$$

such that : $xyz = 1$, " = " iff $x = y = z = 1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $xyz = 1$, we may put $x=\frac{bc}{a^2}$, $y=\frac{ca}{b^2}$, $z=\frac{ab}{c^2}$, where a, b, c are some positive real numbers. Then, the desired inequality becomes

$$\frac{7a^4+a^2bc}{(a^2+bc)^2}+\frac{7b^4+ab^2c}{(b^2+ca)^2}+\frac{7c^4+abc^2}{(c^2+ab)^2}\geq 6.$$

By the Cauchy Schwarz Inequality, we have

$$\sum_{cyc}\frac{7a^4+a^2bc}{(a^2+bc)^2}\geq\sum_{cyc}\frac{7a^4+a^2bc}{(a^2+b^2)(a^2+c^2)}=\frac{\sum_{cyc}(7a^4+a^2bc)(b^2+c^2)}{(a^2+b^2)(b^2+c^2)(a^2+c^2)}\overset{?}{\geq}6$$

$$\Leftrightarrow\sum_{cyc}(7a^4+a^2bc)(b^2+c^2)\geq 6(a^2+b^2)(b^2+c^2)(a^2+c^2)$$

$$\Leftrightarrow\sum_{cyc}a^4(b^2+c^2)+abc\sum_{cyc}a(b^2+c^2)\geq 12a^2b^2c^2 \qquad (1)$$

which is true by AM − GM inequality:

$$LHS_{(1)}\geq\sum_{cyc}a^4.2bc+abc\sum_{cyc}a.2bc\geq 2abc.3abc+6a^2b^2c^2=12a^2b^2c^2.$$

Equality holds iff $x=y=z=1$.