

# ROMANIAN MATHEMATICAL MAGAZINE

If  $\{x, y, z\}$  be non – negative real numbers such that :  $x + y + z = 3$ ,  
 then prove that :  $\frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$

*Proposed by Shirvan Tahirov-Azerbaijan*

*Solution by Soumava Chakraborty-Kolkata-India*

**Case 1** Exactly 1 variable equals to zero and WLOG we may assume

$$x = 0 \text{ (} y + z = 3 \text{ with } y, z > 0 \text{) and then : } \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$$

$$\Leftrightarrow \frac{6}{1 + yz} + 9 \leq 4((y + z)^2 - 2yz) \stackrel{y+z=3}{\Leftrightarrow} \frac{6}{1 + t} + 9 \leq 4(9 - 2t) \text{ (} t = yz \text{)}$$

$$\Leftrightarrow 8t^2 - 19t - 21 \geq 0 \Leftrightarrow t \geq \frac{19 - \sqrt{1033}}{16} \approx -0.82127 \text{ and } t \leq \frac{19 + \sqrt{1033}}{16}$$

$$\approx 3.19627 \rightarrow \text{true} \because t \in \left(0, \frac{9}{4}\right] \left(\because 3 = y + z \stackrel{A-G}{\geq} 2\sqrt{yz} \Rightarrow \sqrt{t} \leq \frac{3}{2} \Rightarrow t \leq \frac{9}{4}\right)$$

$$\therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} < 6$$

**Case 2** Exactly 2 variables equal to zero and WLOG we may assume  $y = z = 0$

$$(x = 3) \text{ and then : } \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6 \Leftrightarrow \frac{12 - 8 \cdot 9}{1} + 12 + 12$$

$$= -36 < 6 \therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} < 6$$

**Case 3**  $x, y, z > 0$  and then :  $\frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$

$$\Leftrightarrow 6 \sum_{\text{cyc}} \frac{1}{1 + yz} \leq 4 \sum_{\text{cyc}} \frac{x^2}{1 + yz} + 3 \rightarrow \text{(i)}$$

$$\sum_{\text{cyc}} \frac{1}{1 + yz} = \frac{\sum_{\text{cyc}}((1 + zx)(1 + xy))}{(1 + xy)(1 + yz)(1 + zx)} = \frac{xyz \sum_{\text{cyc}} x + 2 \sum_{\text{cyc}} xy + 3}{x^2y^2z^2 + xyz \sum_{\text{cyc}} x + \sum_{\text{cyc}} xy + 1}$$

$$\stackrel{x+y+z=3}{=} \frac{3xyz + 2 \sum_{\text{cyc}} xy + 3}{x^2y^2z^2 + 3xyz + \sum_{\text{cyc}} xy + 1} \Rightarrow 6 \sum_{\text{cyc}} \frac{1}{1 + yz} - 3$$

$$= \frac{18xyz + 12 \sum_{\text{cyc}} xy + 18}{x^2y^2z^2 + 3xyz + \sum_{\text{cyc}} xy + 1} - 3$$

$$\Rightarrow 6 \sum_{\text{cyc}} \frac{1}{1 + yz} - 3 \stackrel{(*)}{=} \frac{9xyz + 9 \sum_{\text{cyc}} xy + 15 - 3x^2y^2z^2}{x^2y^2z^2 + 3xyz + \sum_{\text{cyc}} xy + 1} \text{ and}$$

$$4 \sum_{\text{cyc}} \frac{x^2}{1 + yz} = \frac{4 \sum_{\text{cyc}} (x^2(1 + zx)(1 + xy))}{(1 + xy)(1 + yz)(1 + zx)}$$

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$$\begin{aligned}
 &= \frac{4xyz \sum_{\text{cyc}} x^3 + 4(\sum_{\text{cyc}} x^2)(\sum_{\text{cyc}} xy) - 4xyz \sum_{\text{cyc}} x + 4 \sum_{\text{cyc}} x^2}{x^2 y^2 z^2 + xyz \sum_{\text{cyc}} x + \sum_{\text{cyc}} xy + 1} \\
 &\stackrel{x+y+z=3}{=} \frac{4xyz \sum_{\text{cyc}} x^3 + 4(\sum_{\text{cyc}} x^2)(\sum_{\text{cyc}} xy) - 12xyz + 4 \sum_{\text{cyc}} x^2}{x^2 y^2 z^2 + 3xyz + \sum_{\text{cyc}} xy + 1} \\
 \therefore 4 \sum_{\text{cyc}} \frac{x^2}{1+yz} &\stackrel{(**)}{=} \frac{4xyz \sum_{\text{cyc}} x^3 + 4(\sum_{\text{cyc}} x^2)(\sum_{\text{cyc}} xy) - 12xyz + 4 \sum_{\text{cyc}} x^2}{x^2 y^2 z^2 + 3xyz + \sum_{\text{cyc}} xy + 1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore (*), (**) \Rightarrow (i) &\Leftrightarrow 4xyz \sum_{\text{cyc}} x^3 + 4 \left( \sum_{\text{cyc}} x^2 \right) \left( \sum_{\text{cyc}} xy \right) - 12xyz + 4 \sum_{\text{cyc}} x^2 \\
 &\geq 9xyz + 9 \sum_{\text{cyc}} xy + 15 - 3x^2 y^2 z^2 \stackrel{x+y+z=3}{\Leftrightarrow}
 \end{aligned}$$

$$\begin{aligned}
 &4xyz \sum_{\text{cyc}} x^3 + \frac{4}{9} \cdot \left( \sum_{\text{cyc}} x^2 \right) \left( \sum_{\text{cyc}} xy \right) \left( \sum_{\text{cyc}} x \right)^2 - \frac{12xyz}{27} \cdot \left( \sum_{\text{cyc}} x \right)^3 \\
 &+ \frac{4}{81} \left( \sum_{\text{cyc}} x^2 \right) \left( \sum_{\text{cyc}} x \right)^4 \geq \frac{9xyz}{27} \cdot \left( \sum_{\text{cyc}} x \right)^3 \\
 &+ \frac{9}{81} \cdot \left( \sum_{\text{cyc}} xy \right) \left( \sum_{\text{cyc}} x \right)^4 + \frac{15}{729} \cdot \left( \sum_{\text{cyc}} x \right)^6 - 3x^2 y^2 z^2
 \end{aligned}$$

$$\Leftrightarrow \boxed{
 \begin{aligned}
 &972xyz \sum_{\text{cyc}} x^3 + 108 \left( \sum_{\text{cyc}} x^2 \right) \left( \sum_{\text{cyc}} xy \right) \left( \sum_{\text{cyc}} x \right)^2 - 189xyz \left( \sum_{\text{cyc}} x \right)^3 + \\
 &12 \left( \sum_{\text{cyc}} x^2 \right) \left( \sum_{\text{cyc}} x \right)^4 \stackrel{(*)}{\geq} 27 \left( \sum_{\text{cyc}} xy \right) \left( \sum_{\text{cyc}} x \right)^4 + 5 \left( \sum_{\text{cyc}} x \right)^6 - 729x^2 y^2 z^2
 \end{aligned}
 }$$

Assigning  $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z > 0, b+c-a=2x > 0$  and  $c+a-b=2y > 0 \Rightarrow a+b > c, b+c > a, c+a > b \Rightarrow a, b, c$  form sides of a triangle with semiperimeter, circumradius and inradius =  $s, R, r$  (say)

$$\text{yielding } 2 \sum_{\text{cyc}} x = \sum_{\text{cyc}} a = 2s \Rightarrow \sum_{\text{cyc}} x \stackrel{(1)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$$

$$\Rightarrow xyz \stackrel{(2)}{=} r^2 s$$

$$\text{Via such substitutions, } \sum_{\text{cyc}} xy = \sum_{\text{cyc}} (s-a)(s-b) = 4Rr + r^2$$

$$\Rightarrow \sum_{\text{cyc}} xy \stackrel{(3)}{=} 4Rr + r^2 \Rightarrow \sum_{\text{cyc}} x^2 = \left( \sum_{\text{cyc}} x \right)^2 - 2 \sum_{\text{cyc}} xy \stackrel{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{\text{cyc}} x^2 \stackrel{(4)}{=} s^2 - 8Rr - 2r^2 \text{ and}$$

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$$\sum_{\text{cyc}} x^3 = \left( \sum_{\text{cyc}} x \right)^3 - 3(x+y)(y+z)(z+x) \stackrel{\text{via (1)}}{=} s^3 - 3 \cdot 4Rr s$$

$$\Rightarrow \sum_{\text{cyc}} x^3 \stackrel{(5)}{=} s(s^2 - 12Rr) \therefore \text{via (1), (2), (3), (4) and (5), } (\bullet) \Leftrightarrow$$

$$972r^2s^2(s^2 - 12Rr) + 108(s^2 - 8Rr - 2r^2)(4Rr + r^2)s^2 - 189r^2s^4$$

$$+ 12(s^2 - 8Rr - 2r^2)s^4 \geq 27(4Rr + r^2)s^4 + 5s^6 - 729r^4s^2$$

$$\Leftrightarrow \boxed{7s^4 + rs^2(228R + 840r) \stackrel{(\bullet\bullet)}{\geq} r(3456R^2 + 13392Rr - 513r^2)}$$

Now, LHS of  $(\bullet\bullet) \stackrel{\text{Gerretsen}}{\geq} (7(16Rr - 5r^2) + r(228R + 840r))s^2 \stackrel{\text{Gerretsen}}{\geq}$

$$(340Rr + 805r^2)(16Rr - 5r^2) \stackrel{?}{\geq} r(3456R^2 + 13392Rr - 513r^2)$$

$$\Leftrightarrow 496R^2 - 553Rr - 878r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R - 2r)(496R + 439r) \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$\therefore R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (\bullet\bullet) \Rightarrow (\bullet)$  is true  $\therefore \frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$  and

combining all cases,  $\frac{12 - 8x^2}{1 + yz} + \frac{12 - 8y^2}{1 + xz} + \frac{12 - 8z^2}{1 + xy} \leq 6$

$\forall$  non - negative  $\{x, y, z\}$  such that :  $x + y + z = 3,$  " = " iff  $x = y = z = 1$  (QED)