ROMANIAN MATHEMATICAL MAGAZINE

If $\{x,y,z\}\in\mathbb{R}^+$ such that : $xyz=1$, then prove that :

$$\frac{x}{x^2+2}+\frac{y}{y^2+2}+\frac{z}{z^2+2}\le 1$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{x^2+2}+\frac{y}{y^2+2}+\frac{z}{z^2+2}=\frac{\sum_{cyc}\left(x(y^2+2)(z^2+2)\right)}{(x^2+2)(y^2+2)(z^2+2)}$$

$$=\frac{\sum_{cyc}\left(x(2y^2+2z^2+y^2z^2+4)\right)}{x^2y^2z^2+8+2\sum_{cyc}x^2y^2+4\sum_{cyc}x^2}$$

$$=\frac{2(\sum_{cyc}x)(\sum_{cyc}xy)-6xyz+xyz\sum_{cyc}xy+4\sum_{cyc}x}{x^2y^2z^2+8+2\sum_{cyc}x^2y^2+4\sum_{cyc}x^2}\le 1$$

$$\overset{xyz=1}{\Leftrightarrow}\boxed{15+2\sum_{cyc}x^2y^2+4\sum_{cyc}x^2\overset{(*)}{\ge}2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+\sum_{cyc}xy+4\sum_{cyc}x}$$

Now, $x^2+x^2y^2\overset{A-G}{\ge}2x^2y,\ y^2+y^2z^2\overset{A-G}{\ge}2y^2z,\ z^2+z^2x^2\overset{A-G}{\ge}2z^2x,$

$y^2+x^2y^2\overset{A-G}{\ge}2xy^2,\ z^2+y^2z^2\overset{A-G}{\ge}2yz^2,\ x^2+z^2x^2\overset{A-G}{\ge}2zx^2$

and summing up, we *arrive at* :

$$2\sum_{cyc}x^2+2\sum_{cyc}x^2y^2\ge 2\sum_{cyc}x^2y+2\sum_{cyc}xy^2=2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6xyz$$

$$\therefore 2\sum_{cyc}x^2+2\sum_{cyc}x^2y^2+6\ge 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\to\text{(i)}\ \therefore\ \text{(i)}\Rightarrow\text{in order}$$

to prove $(*)$, it suffices to prove : $9+2\sum_{cyc}x^2\ge\sum_{cyc}xy+4\sum_{cyc}x$

$$\overset{xyz=1}{\Leftrightarrow}\boxed{2\sum_{cyc}x^2-\sum_{cyc}xy+9.\sqrt[3]{x^2y^2z^2}\overset{(**)}{\ge}4\left(\sum_{cyc}x\right).\sqrt[3]{xyz}}$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ *and* $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of *a* triangle with semiperimeter, circumr*a*dius *and* inr*a*dius $=s,R,r$ (*say*)

yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(1)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$

$\Rightarrow xyz\overset{(2)}{=}r^2s$ *and* vi*a* such substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)$

$$=4Rr+r^2\Rightarrow\sum_{cyc}xy\overset{(3)}{=}4Rr+r^2\Rightarrow\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy$$

$$\overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc} x^2 \overset{(4)}{=} s^2-8Rr-2r^2$$

$$\therefore \text{via } (1),(2),(3) \text{ and } (4), (**) \Leftrightarrow 2(s^2-8Rr-2r^2)-4Rr-r^2+9.\sqrt[3]{r^4s^2}$$

$$\geq 4s.\sqrt[3]{r^2s} \Leftrightarrow \boxed{\left(2s^2-20Rr-5r^2+9.\sqrt[3]{r^4s^2}\right)^3 \overset{(***)}{\geq} 64s^4r^2}$$

$$\text{Now, } \left(2s^2-20Rr-5r^2+9.\sqrt[3]{r^4s^2}\right)^3$$

$$=(2s^2-20Rr-5r^2)^3+729r^4s^2$$

$$+27.\sqrt[3]{r^4s^2}.(2s^2-20Rr-5r^2)\left(2s^2-20Rr-5r^2+9.\sqrt[3]{r^4s^2}\right)$$

$$\overset{\text{Mitrinovic}}{\geq} (2s^2-20Rr-5r^2)^3+729r^4s^2$$

$$+81r^2(2s^2-20Rr-5r^2)(2s^2-20Rr-5r^2+27r^2) \overset{?}{\geq} 64s^4r^2$$

$$\Leftrightarrow \boxed{\begin{gathered} 8s^6-(240Rr-200r^2)s^4+r^2s^2(2400R^2-5280Rr+3633r^2)- \\ r^3(8000R^3-26400R^2r+29040Rr^2+9035r^3) \underset{(****)}{\overset{?}{\geq}} 0 \end{gathered}} \text{ and}$$

$$\because 8(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0, \therefore \text{ in order to prove } (****), \text{it suffices to prove :}$$

$$\text{LHS of } (****) \geq 8(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (144Rr+80r^2)s^4-r^2s^2(3744R^2+1440Rr-3033r^2)$$

$$+r^3(24768R^3-4320R^2r-19440Rr^2-10035r^3) \overset{(*****)}{\geq} 0 \text{ and}$$

$$\because (144Rr+80r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0, \therefore \text{ in order to prove } (*****),$$

$$\text{it suffices to prove : LHS of } (*****) \geq (144Rr+80r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (864R^2-320Rr+2233r^2)s^2 \overset{(******)}{\geq} r\begin{pmatrix} 12096R^3+1760R^2r \\ +10240Rr^2+12035r^3 \end{pmatrix} \text{ and finally,}$$

$$(864R^2-320Rr+2233r^2)s^2 \overset{\text{Gerretsen}}{\geq} (864R^2-320Rr+2233r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(12096R^3+1760R^2r+10240Rr^2+12035r^3)$$

$$\Leftrightarrow 108t^3-700t^2+1693t-1450 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \boxed{\boxed{(t-2)(108t^2-484t+725) \overset{?}{\geq} 0}} \rightarrow \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \text{ and discriminant of } (108t^2-484t+725)=484^2-432*725$$

$$=-78944 \Rightarrow 108t^2-484t+725>0 \text{ and so,}$$

$$(******) \Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \frac{x}{x^2+2}+\frac{y}{y^2+2}+\frac{z}{z^2+2} \leq 1 \ \forall \{x,y,z\} \in \mathbb{R}^+ \text{ such that : } xyz=1,$$

$$''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{x}{x^2+2}+\frac{y}{y^2+2}+\frac{z}{z^2+2}=\frac{x}{x^2+1+1}+\frac{y}{y^2+1+1}+\frac{z}{z^2+1+1}$$

$$\overset{AM-GM}{\leq}\frac{x}{2x+1}+\frac{y}{2y+1}+\frac{z}{2z+1}$$

$$=\left(\frac{1}{2}-\frac{1}{2(2x+1)}\right)+\left(\frac{1}{2}-\frac{1}{2(2y+1)}\right)+\left(\frac{1}{2}-\frac{1}{2(2z+1)}\right)$$

$$=\frac{3}{2}-\frac{1}{2}\left(\frac{yz}{2+yz}+\frac{zx}{2+zx}+\frac{xy}{2+xy}\right)\overset{CBS}{\leq}\frac{3}{2}-\frac{\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right)^2}{2(6+yz+zx+xy)}$$

$$=\frac{3}{2}-\frac{xy+yz+zx+2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2(6+xy+yz+zx)}\overset{AM-GM}{\leq}\frac{3}{2}-\frac{xy+yz+zx+2.3\sqrt[6]{xyz}}{2(6+xy+yz+zx)}=$$

$$=\frac{3}{2}-\frac{1}{2}=1$$

Equality holds iff $x=y=z=1$.