ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $x + y + z = 3$ then:

$$\frac{xy}{\sqrt{z^2+8}} + \frac{xz}{\sqrt{y^2+8}} + \frac{yz}{\sqrt{x^2+8}} \le 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Mirsadix Muzefferov-Azerbaijan

$$\begin{cases} \dfrac{xy}{\sqrt{z^2+8}} = \dfrac{xy}{\sqrt{z^2+1+7}} \overset{(1)}{\underset{(2)}{\le}} \dfrac{\frac{1}{4}(x+y)^2}{\sqrt{2z+7}} = \dfrac{(x+y)^2}{4\sqrt{2z+7}} \\ \dfrac{xz}{\sqrt{y^2+8}} = \dfrac{xz}{\sqrt{y^2+1+7}} \overset{(1)}{\underset{(2)}{\le}} \dfrac{\frac{1}{4}(x+z)^2}{\sqrt{2y+7}} = \dfrac{(x+z)^2}{4\sqrt{2y+7}} \\ \dfrac{yz}{\sqrt{x^2+8}} = \dfrac{yz}{\sqrt{x^2+1+7}} \overset{(1)}{\underset{(2)}{\le}} \dfrac{\frac{1}{4}(y+z)^2}{\sqrt{2x+7}} = \dfrac{(y+z)^2}{4\sqrt{2x+7}} \end{cases}$$

$Because,\ x + y \ge 2\sqrt{xy} \Rightarrow xy \le \frac{1}{4}(x+y)^2\ (1) \quad z^2 + 1 \ge 2z\ (2)$

$$\frac{xy}{\sqrt{z^2+8}} + \frac{xz}{\sqrt{y^2+8}} + \frac{yz}{\sqrt{x^2+8}} \le \frac{1}{4}\left(\frac{(x+y)^2}{\sqrt{2z+7}} + \frac{(x+z)^2}{\sqrt{2y+7}} + \frac{(y+z)^2}{\sqrt{2x+7}}\right)$$

$$\le \frac{1}{4}\frac{\left((x+y)+(x+z)+(y+z)\right)^2}{3^{\frac{1}{2}}(2z+7+2y+7+2x+7)^{\frac{1}{2}}} = \frac{1}{4}\cdot\frac{6^2}{3^{\frac{1}{2}}(27)^{\frac{1}{2}}} = \frac{1}{4}\cdot\frac{36}{9} = 1$$

$Equality\ holds\ iff\ x = y = z = 1$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{yz}{\sqrt{x^2+8}} = \frac{3yz}{\sqrt{(1+8)(x^2+8)}} \le \frac{3yz}{x+8} = \frac{3yz}{(z+x)+(x+y)+5}$$
$$\le \frac{3yz}{(2+2+5)^2}\left(\frac{4}{z+x} + \frac{4}{x+y} + 5\right) = \frac{4}{27}\left(\frac{yz}{z+x} + \frac{yz}{x+y}\right) + \frac{5yz}{27}.$$

Similarly, we have

$$\frac{xy}{\sqrt{z^2+8}} \le \frac{4}{27}\left(\frac{xy}{y+z} + \frac{xy}{z+x}\right) + \frac{5xy}{27} \text{ and } \frac{zx}{\sqrt{y^2+8}} \le \frac{4}{27}\left(\frac{zx}{x+y} + \frac{zx}{y+z}\right) + \frac{5zx}{27}.$$

Adding these inequalities, we obtain

$$\frac{xy}{\sqrt{z^2+8}}+\frac{yz}{\sqrt{x^2+8}}+\frac{zx}{\sqrt{y^2+8}}\le\frac{4}{27}\left(\frac{xy+zx}{y+z}+\frac{yz+xy}{z+x}+\frac{yz+zx}{x+y}\right)+\frac{5(xy+yz+zx)}{27}$$

$$\le\frac{4}{27}(x+y+z)+\frac{5(x+y+z)^2}{3.27}=\frac{4}{27}.3+\frac{5.3^2}{3.27}=1.$$

Equality holds iff $x=y=z=1$.