ROMANIAN MATHEMATICAL MAGAZINE

Let $\{x, y, z\}$ be positive real nmbers such that : $x + y + z = 3$. Prove that :

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \le \frac{3\sqrt{2}}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} =$$

$$= \frac{1}{\sqrt{(x+y)(y+z)(z+x)}} \cdot \sum_{cyc}\left(\sqrt{x(y+z)} \cdot \sqrt{z+x}\right) \overset{CBS}{\le}$$

$$\frac{1}{\sqrt{(x+y)(y+z)(z+x)}} \cdot \sqrt{\sum_{cyc} x(y+z)}\sqrt{\sum_{cyc}(z+x)} = \frac{\sqrt{2\sum_{cyc} xy} \cdot \sqrt{2\sum_{cyc} x}}{\sqrt{(\sum_{cyc} x)(\sum_{cyc} xy) - xyz}}$$

$$\overset{?}{\le} \frac{3}{\sqrt{2}} \Leftrightarrow 9\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz \overset{?}{\ge} 8\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{?}{\ge} 9xyz \to \text{true} \because \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{A-G}{\ge} 3.\sqrt[3]{xyz}.3.\sqrt[3]{x^2y^2z^2}$$

$$= 9xyz \therefore \sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \le \frac{3\sqrt{2}}{2} \forall x, y, z > 0, '' = '' \text{ iff } x = y = z \text{ (QED)}$$

Solution 2 by Pham Duc Nam-Vietnam

$x, y, z > 0 \quad x + y + z = 3$

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \le \frac{3\sqrt{2}}{2} \ (1)? \quad * (1) <=> \sqrt{\frac{2x}{x+y}} + \sqrt{\frac{2y}{y+z}} + \sqrt{\frac{2z}{z+x}} \le 3$$

$We\ will\ prove\ this\ inequality\ is\ true\ for\ any\ x, y, z > 0$

$$* \sqrt{\frac{2x}{x+y}} + \sqrt{\frac{2y}{y+z}} + \sqrt{\frac{2z}{z+x}} = \sqrt{\frac{2x(x+z)}{(x+y)(x+z)}} + \sqrt{\frac{2y(y+x)}{(y+z)(y+x)}} + \sqrt{\frac{2z(z+y)}{(x+z)(y+z)}}$$

$$\le \sqrt{(2x+2y+2z)\left(\frac{2x}{(x+y)(x+z)} + \frac{2y}{(y+z)(y+x)} + \frac{2z}{(x+z)(y+z)}\right)}$$

$So, we\ only\ need\ to\ prove:$

$$(2x+2y+2z)\left(\frac{2x}{(x+y)(x+z)}+\frac{2y}{(y+z)(y+x)}+\frac{2z}{(x+z)(y+z)}\right)\leq$$

$<=> 4(x+y+z)\big(x(y+z)+y(x+z)+z(x+y)\big)\leq 9(x+y)(y+z)(x+z)$

$<=> 8(x+y+z)(xy+yz+xz)\leq 9(x+y)(y+z)(x+z)$

$<=> \ (x+y)(y+z)(x+z)\geq \frac{8}{9}(x+y+z)(xy+yz+xz)\geq \frac{8}{9}\left(\sqrt[3]{xyz}\right)\left(\sqrt[3]{x^2y^2z^2}\right)=8xyz$

which is true by AM − GM

$<=> \ \sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}}\leq 3\ \forall\ x,y,z>0$

$<=> \ \sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}\leq\frac{3\sqrt{2}}{2}$, *equality holds if and only* $x=y=z$

In this case : $x+y+z=3 \ =>$ *equality holds if and only* $x=y=z=1$