ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$, then :

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca) + \left(2\sqrt[3]{abc} + 1\right)\left(\sqrt[3]{abc} - 1\right)^2$$

Proposed by Sidi Abdullah Lemrabott-Mauritania
Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ $a = b = c = 0$ and then : $LHS - RHS = 1 - 1 \Rightarrow LHS = RHS$
$\boxed{\text{Case 2}}$ Exactly 2 variables $= 0$ and WLOG we may assume $b = c = 0$ $(a > 0)$ and then : $LHS - RHS = a^2 + 1 - 1 > 0 \Rightarrow LHS > RHS$
$\boxed{\text{Case 3}}$ Exactly 1 variable $= 0$ and WLOG we may assume $a = 0$ $(b, c > 0)$ and then : $LHS - RHS = b^2 + c^2 + 1 - 2bc - 1 = (b - c)^2 \geq 0 \Rightarrow LHS \geq RHS$
$\boxed{\text{Case 4}}$ $a, b, c > 0$ and let $\sqrt[3]{a} = x, \sqrt[3]{b} = y, \sqrt[3]{c} = z$ and then :

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca) + \left(2\sqrt[3]{abc} + 1\right)\left(\sqrt[3]{abc} - 1\right)^2$$

$$\Leftrightarrow \sum_{cyc} x^6 + 3x^2y^2z^2 \geq 2\sum_{cyc} x^3y^3$$

$$\Leftrightarrow 3x^2y^2z^2 + \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x^4 - \sum_{cyc} x^2y^2\right) + 3x^2y^2z^2$$

$$\geq 6x^2y^2z^2 + 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2y^2 - xyz\sum_{cyc} x\right)$$

$$\Leftrightarrow \boxed{\left(\sum_{cyc} x^2\right)\left(\left(\sum_{cyc} x^2\right)^2 - 3\sum_{cyc} x^2y^2\right) \overset{(*)}{\geq} 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2y^2 - xyz\sum_{cyc} x\right)}$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \rightarrow (1) \Rightarrow x = s - X, y = s - Y,$ $z = s - Z$ and such substitutions $\Rightarrow \sum_{cyc} xy = \sum_{cyc}(s - X)(s - Y)$

$$\Rightarrow \sum_{cyc} xy = 4Rr + r^2 \rightarrow (2), \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy$$

$$\overset{\text{via (1) and (2)}}{=} s^2 - \left(4Rr + r^2\right) \Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \rightarrow (3) \text{ and}$$

$$xyz = (s - X)(s - Y)(s - Z) = r^2s \rightarrow (4) \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x$$

$$\overset{\text{via }(1),(2)\text{ and }(4)}{=} \left(4Rr+r^2\right)^2-2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 = \left(4Rr+r^2\right)^2-2r^2s^2 \to (5)$$

$$\therefore (*) \Leftrightarrow \left(s^2-8Rr-2r^2\right)\left(\left(s^2-8Rr-2r^2\right)^2-3\left(\left(4Rr+r^2\right)^2-2r^2s^2\right)\right)$$

$$\geq \left(4Rr+r^2\right)\left(\left(4Rr+r^2\right)^2\right.$$

$$\left.-3r^2s^2\right) \boxed{s^6-24Rrs^4+r^2s^2\left(144R^2+48Rr+3r^2\right)-4r^3(4R+r)^3 \overset{(**)}{\geq} 0} \text{ and}$$

$\because \left(s^2-16Rr+5r^2\right)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ **in order to prove** $(**)$, **it suffices to prove :**

LHS of $(**) \geq \left(s^2-16Rr+5r^2\right)^3$

$$\Leftrightarrow \left(24Rr-15r^2\right)s^4-r^2s^2\left(624R^2-528Rr+72r^2\right)$$

$$+r^3\left(3840R^3-4032R^2r+1152Rr^2-129r^3\right) \overset{(***)}{\geq} 0 \text{ and}$$

$\because \left(24Rr-15r^2\right)\left(s^2-16Rr+5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ **in order to prove** $(***)$,

it suffices to prove : LHS of $(***) \geq \left(24Rr-15r^2\right)\left(s^2-16Rr+5r^2\right)^2$

$$\Leftrightarrow \left(24R^2-32Rr+13r^2\right)s^2 \overset{(****)}{\geq} r\left(384R^3-608R^2r+308Rr^2-41r^3\right)$$

Now, LHS of $(****) \overset{\text{Rouche}}{\geq} \left(24R^2-32Rr+13r^2\right)\begin{pmatrix} 2R^2+10Rr-r^2 \\ -2(R-2r).\sqrt{R^2-2Rr} \end{pmatrix}$

$$\overset{?}{\geq} r\left(384R^3-608R^2r+308Rr^2-41r^3\right)$$

$$\Leftrightarrow \boxed{(R-2r)\left(24R^3-56R^2r+33Rr^2-7r^3\right) \underset{(*****)}{\overset{?}{\geq}} (R-2r)\left(24R^2-32Rr+13r^2\right).\sqrt{R^2-2Rr}}$$

Now, $24R^3-56R^2r+33Rr^2-7r^3 = (R-2r)\left(24R^2-8Rr+17r^2\right)+27r^3 \overset{\text{Euler}}{\geq}$

$27r^3 > 0$ *and* $\because R-2r \overset{\text{Euler}}{\geq} 0 \therefore$ **in order to prove** $(*****)$, **it suffices to prove :**

$$24R^3-56R^2r+33Rr^2-7r^3 > \left(24R^2-32Rr+13r^2\right).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow \left(24R^3-56R^2r+33Rr^2-7r^3\right)^2 > \left(R^2-2Rr\right)\left(24R^2-32Rr+13r^2\right)^2$$

$$\Leftrightarrow 96R^3+40R^2r-124Rr^2+49r^3 > 0$$

$\Leftrightarrow \boxed{65R^3+40R^2r+31R\left(R^2-4r^2\right)+49r^3>0} \to$ **true** $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*****)$

$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ **is true** $\therefore a^2+b^2+c^2+2abc+1 \geq$

$2(ab+bc+ca)+\left(2\sqrt[3]{abc}+1\right)\left(\sqrt[3]{abc}-1\right)^2 \;\forall\, a,b,c>0 \therefore$ *combining all* **cases,**

$$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ca)+\left(2\sqrt[3]{abc}+1\right)\left(\sqrt[3]{abc}-1\right)^2$$

$\forall\, a,b,c \geq 0,'' ='' $ **iff** $(a=b=c)$ **or** $(a=0, b=c\neq 0)$ **or** $(b=0, c=a\neq 0)$
or $(c=0, a=b\neq 0)$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows

$$a^2+b^2+c^2+3\sqrt[3]{(abc)^2}\geq 2(ab+bc+ca).$$

By Schur's inequality on the triple $\left(\sqrt[3]{a^2},\sqrt[3]{b^2},\sqrt[3]{c^2}\right)$, **we have**

$$a^2+b^2+c^2+3\sqrt[3]{(abc)^2}$$
$$\geq \sqrt[3]{a^2b^2}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)+\sqrt[3]{b^2c^2}\left(\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)+\sqrt[3]{c^2a^2}\left(\sqrt[3]{c^2}+\sqrt[3]{a^2}\right).$$

Also, by AM − GM inequality, we have

$$\sqrt[3]{a^2}+\sqrt[3]{b^2}\geq 2\sqrt[3]{ab}\ \ \text{(and analogs)}.$$

Therefore

$$a^2+b^2+c^2+3\sqrt[3]{(abc)^2}\geq 2(ab+bc+ca).$$

Equality holds iff $(a=b=c)$ **or** $(a=0, b=c)$ **and permutation.**