ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c, x, y, z \in \mathbb{R}$, then :

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\ge\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}$$

When does equality hold ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}\sqrt{b^2+c^2}\right)^2}{2\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}(b^2+c^2)+2\sum_{cyc}\left(\sqrt{b^2+c^2}.\sqrt{c^2+a^2}\right)}{2\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}(b^2+c^2)(c^2+a^2)+2\sum_{cyc}\left((c^2+a^2).\sqrt{b^2+c^2}.\sqrt{a^2+b^2}\right)}}{\sum_{cyc}x^2}$$

$$\ge\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+2\sum_{cyc}\left((c^2+a^2)\left(\frac{b+c}{\sqrt2}\right)\left(\frac{a+b}{\sqrt2}\right)\right)}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+\sum_{cyc}\left((c^2+a^2)\left(\sum_{cyc}ab+b^2\right)\right)}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+2\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+3\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$\ge\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$\Rightarrow\boxed{\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\overset{(*)}{\ge}\frac{m+|m+n|}{\sum_{cyc}x^2}}\left(m=\sum_{cyc}a^2\ \text{and}\ n=\sum_{cyc}ab\right)$$

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$$\text{Again, } \boxed{\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A} = \frac{\left(\sum_{cyc} a\right)\left(\sqrt{3\sum_{cyc} a^2}\right)}{2\sum_{cyc} x^2}$$

$$+\frac{3\sum_{cyc} a^2+\left(\sum_{cyc} a\right)^2-2\left(\sum_{cyc} a\right)\left(\sqrt{3\sum_{cyc} a^2}\right)}{2\sum_{cyc} x^2}$$

$$\left(\text{where : } A=\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}\right)=\frac{4\sum_{cyc} a^2+2\sum_{cyc} ab}{2\sum_{cyc} x^2}$$

$$\boxed{=\frac{2m+n}{\sum_{cyc} x^2}\le\frac{m+|m+n|}{\sum_{cyc} x^2}}\ (\because |t|\ge t)\overset{\text{via } (*)}{\le}\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}$$

$$\therefore \frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\ge\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}$$

$$+\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}$$

$$\ge\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)},\ ''=''\ \text{ iff } (a=b=c=0)$$

$$\text{or } (a=b=c\ne 0 \text{ and } |x|=|y|=|z|)\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}\ge\frac{\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2}{(x^2+y^2)+(y^2+z^2)+(z^2+x^2)}$$

$$=\frac{a^2+b^2+c^2+\sqrt{(a^2+b^2)(a^2+c^2)}+\sqrt{(b^2+c^2)(b^2+a^2)}+\sqrt{(c^2+a^2)(c^2+b^2)}}{x^2+y^2+z^2}$$

$$\ge\frac{a^2+b^2+c^2+(a^2+bc)+(b^2+ca)+(c^2+ab)}{x^2+y^2+z^2}=\frac{(a+b+c)^2+3(a^2+b^2+c^2)}{x^2+y^2+z^2}$$

$$\overset{AM-GM}{\ge}\frac{2(a+b+c)\sqrt{3(a^2+b^2+c^2)}}{2(x^2+y^2+z^2)}=\left(\frac{a+b+c}{x^2+y^2+z^2}\right)\sqrt{3(a^2+b^2+c^2)}.$$

as desired. Equality holds iff $(a=b=c=0)$ **or** $(a=b=c\ne 0 \text{ and } x^2=y^2=z^2)$.