ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c, x, y, z \in \mathbb{R}$, then :

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2} \ge \left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A,$$

$$\text{where : } A=\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}$$

When does equality holds ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{cyc}\sqrt{b^2+c^2}\right)^2}{2\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}(b^2+c^2)+2\sum_{cyc}\left(\sqrt{b^2+c^2}.\sqrt{c^2+a^2}\right)}{2\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}(b^2+c^2)(c^2+a^2)+2\sum_{cyc}\left((c^2+a^2).\sqrt{b^2+c^2}.\sqrt{a^2+b^2}\right)}}{\sum_{cyc}x^2}$$

$$\ge\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+2\sum_{cyc}\left((c^2+a^2)\left(\frac{b+c}{\sqrt{2}}\right)\left(\frac{a+b}{\sqrt{2}}\right)\right)}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+\sum_{cyc}\left((c^2+a^2)(\sum_{cyc}ab+b^2)\right)}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+2(\sum_{cyc}ab)(\sum_{cyc}a^2)+2\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2(\sum_{cyc}ab)(\sum_{cyc}a^2)+3\sum_{cyc}a^2b^2}}{\sum_{cyc}x^2}$$

$$\ge\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2\right)^2+2(\sum_{cyc}ab)(\sum_{cyc}a^2)+\left(\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$=\frac{\sum_{cyc}a^2+\sqrt{\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)^2}}{\sum_{cyc}x^2}$$

$$\Rightarrow \boxed{\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2} \overset{(*)}{\geq} \frac{m+|m+n|}{\sum_{cyc} x^2}} \left(m=\sum_{cyc} a^2 \text{ and } n=\sum_{cyc} ab\right)$$

$$\text{Again, } \boxed{\left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A} = \frac{\left(\sum_{cyc} a\right)\left(\sqrt{3\sum_{cyc} a^2}\right)}{2\sum_{cyc} x^2}$$

$$+\frac{3\sum_{cyc} a^2+\left(\sum_{cyc} a\right)^2-2\left(\sum_{cyc} a\right)\left(\sqrt{3\sum_{cyc} a^2}\right)}{2\sum_{cyc} x^2}$$

$$=\frac{4\sum_{cyc} a^2+2\sum_{cyc} ab}{2\sum_{cyc} x^2}\boxed{=\frac{2m+n}{\sum_{cyc} x^2} \leq \frac{m+|m+n|}{\sum_{cyc} x^2}}$$

$$(\because |t| \geq t) \overset{\text{via }(*)}{\leq} \frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2} \therefore \frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2}$$

$$\geq \left(\frac{a+b+c}{x^2+y^2+z^2}\right).\sqrt{3(a^2+b^2+c^2)}+A, \text{ where :}$$

$$A=\frac{\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)},$$

$$''='' \text{ iff } (a=b=c=0) \text{ or } (a=b=c\neq 0 \text{ and } |x|=|y|=|z|) \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{a^2+b^2}{x^2+y^2}+\frac{b^2+c^2}{y^2+z^2}+\frac{c^2+a^2}{z^2+x^2} \geq \frac{\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2}{(x^2+y^2)+(y^2+z^2)+(z^2+x^2)}$$

$$=\frac{a^2+b^2+c^2+\sqrt{(a^2+b^2)(a^2+c^2)}+\sqrt{(b^2+c^2)(b^2+a^2)}+\sqrt{(c^2+a^2)(c^2+b^2)}}{x^2+y^2+z^2}$$

$$\geq \frac{a^2+b^2+c^2+(a^2+bc)+(b^2+ca)+(c^2+ab)}{x^2+y^2+z^2}$$

$$=\frac{2(a+b+c)\sqrt{3(a^2+b^2+c^2)}+\left(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)\right)^2}{2(x^2+y^2+z^2)}$$

$$=\left(\frac{a+b+c}{x^2+y^2+z^2}\right)\sqrt{3(a^2+b^2+c^2)}+A,$$

as desired. Equality holds iff $(a=b=c=0)$ **or** $(a=b=c\neq 0$ **and** $x^2=y^2=z^2)$.