ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then : $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \le \frac{3}{2}\cdot\frac{p}{q}\cdot\frac{1}{A}$, where :

$$A = 1+\frac{1}{8pq}\sum_{cyc} c(a-b)^2, p = a+b+c, q = ab+bc+ca$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$\therefore abc = r^2 s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3)$ and we have : $A = 1+\frac{1}{8pq}\cdot\sum_{cyc} c(a-b)^2$

$$= 1+\frac{1}{8(\sum_{cyc} a)(\sum_{cyc} ab)}\cdot\sum_{cyc}\left(c(a^2+b^2-2ab)\right)$$

$$= 1+\frac{1}{8(\sum_{cyc} a)(\sum_{cyc} ab)}\cdot\left(\sum_{cyc}\left(ab\left(\sum_{cyc} a - c\right)\right) - 6abc\right)$$

$$= 1+\frac{(\sum_{cyc} a)(\sum_{cyc} ab) - 9abc}{8(\sum_{cyc} a)(\sum_{cyc} ab)} \overset{\text{via (1),(2) and (3)}}{=} 1+\frac{s(4Rr+r^2)-9r^2 s}{8s(4Rr+r^2)}$$

$$\Rightarrow A = \frac{9R}{2(4R+r)} \therefore \frac{3}{2}\cdot\frac{p}{q}\cdot\frac{1}{A} - \left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \overset{\text{via (1) and (3)}}{=}$$

$$\frac{3}{2}\cdot\frac{s}{4Rr+r^2}\cdot\frac{2(4R+r)}{9R} - \left(\sum_{cyc}\frac{1}{x}\right) = \frac{s}{3Rr} - \frac{s^2+4Rr+r^2}{4Rrs} \overset{?}{\ge} 0$$

$$\Leftrightarrow s^2 - 12Rr - 3r^2 \overset{?}{\ge} 0 \to \text{true} \because s^2 - 12Rr - 3r^2$$

$$= s^2 - 16Rr + 5r^2 + 4r(R-2r) \overset{\text{Gerretsen and Euler}}{\ge} 0 \therefore \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

$$\le \frac{3}{2}\cdot\frac{p}{q}\cdot\frac{1}{A}\ \forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$