ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then :

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{2}$$ When does equality hold ?

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$$\forall a,b,c,x,y,z>0, \frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)}$$
$$=\frac{a^4}{ax(a^2+bc)}+\frac{b^4}{by(b^2+ca)}+\frac{c^4}{cz(c^2+ab)}$$
$$\overset{\text{Bergstrom}}{\geq} \frac{(a^2+b^2+c^2)^2}{ax(a^2+bc)+by(b^2+ca)+cz(c^2+ab)}$$
$$\overset{\text{CBS}}{\geq} \frac{(a^2+b^2+c^2)^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}.\sqrt{(a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2}} \overset{?}{\geq} \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$
$$\Leftrightarrow 4(a^2+b^2+c^2)^2 \overset{?}{\geq} 3\left((a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2\right)$$
$$\Leftrightarrow 4\sum_{cyc}a^4+8\sum_{cyc}a^2b^2 \overset{?}{\geq} 3\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+6abc\sum_{cyc}a$$
$$\Leftrightarrow \sum_{cyc}a^4+5\sum_{cyc}a^2b^2 \overset{?}{\geq} 6abc\sum_{cyc}a \rightarrow \text{true} \because \sum_{cyc}a^4+5\sum_{cyc}a^2b^2 \geq 6\sum_{cyc}a^2b^2$$
$$\geq 6abc\sum_{cyc}a \therefore \frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)} \geq \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$
$$\forall a,b,c,x,y,z>0, '' ='' \text{ iff } (a=b=c \text{ and } x=y=z) \rightarrow (1)$$

Implementing (1) with $x=y=z=1$, we arrive at : $\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}$

$$\geq \frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2.1+b^2.1+c^2.1}} \therefore \frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab} \geq \frac{\sqrt{3(a^2+b^2+c^2)}}{2},$$
$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$