ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then :

$$\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}\ge\frac{a+b+c}{2}$$ When does equality hold ?

Proposed by Sidi Abdullah Lemrabott-Mauritania

Solution by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c,x,y,z>0,\frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)}$$

$$=\frac{a^4}{ax(a^2+bc)}+\frac{b^4}{by(b^2+ca)}+\frac{c^4}{cz(c^2+ab)}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{(a^2+b^2+c^2)^2}{ax(a^2+bc)+by(b^2+ca)+cz(c^2+ab)}$$

$$\overset{\text{CBS}}{\ge}\frac{(a^2+b^2+c^2)^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}.\sqrt{(a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$\Leftrightarrow 4(a^2+b^2+c^2)^2\overset{?}{\ge}3\left((a^2+bc)^2+(b^2+ca)^2+(c^2+ab)^2\right)$$

$$\Leftrightarrow 4\sum_{cyc}a^4+8\sum_{cyc}a^2b^2\overset{?}{\ge}3\sum_{cyc}a^4+3\sum_{cyc}a^2b^2+6abc\sum_{cyc}a$$

$$\Leftrightarrow\sum_{cyc}a^4+5\sum_{cyc}a^2b^2\overset{?}{\ge}6abc\sum_{cyc}a\to\text{true}\because\sum_{cyc}a^4+5\sum_{cyc}a^2b^2\ge 6\sum_{cyc}a^2b^2$$

$$\ge 6abc\sum_{cyc}a\therefore\frac{a^3}{x(a^2+bc)}+\frac{b^3}{y(b^2+ca)}+\frac{c^3}{z(c^2+ab)}\ge\frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$\forall a,b,c,x,y,z>0,''='' \text{ iff } (a=b=c \text{ and } x=y=z)\to(1)$$

Implementing (1) with $x=y=z=1$, we arrive at : $\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}$

$$\ge\frac{\sqrt{3}}{2}\frac{a^2+b^2+c^2}{\sqrt{a^2.1+b^2.1+c^2.1}}=\frac{\sqrt{3(a^2+b^2+c^2)}}{2}\ge\frac{a+b+c}{2}$$

$$\therefore\frac{a^3}{a^2+bc}+\frac{b^3}{b^2+ca}+\frac{c^3}{c^2+ab}\ge\frac{a+b+c}{2},''='' \text{ iff } a=b=c \text{ (QED)}$$