

If  $a, b, c, x, y, z > 0$ , then :

$$\frac{a^3}{x(a^2 + bc)} + \frac{b^3}{y(b^2 + ca)} + \frac{c^3}{z(c^2 + ab)} \geq \frac{\sqrt{3}}{2} \frac{a^2 + b^2 + c^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$$

When does equality hold ?

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$$\begin{aligned} & \frac{a^3}{x(a^2 + bc)} + \frac{b^3}{y(b^2 + ca)} + \frac{c^3}{z(c^2 + ab)} \\ &= \frac{a^4}{ax(a^2 + bc)} + \frac{b^4}{by(b^2 + ca)} + \frac{c^4}{cz(c^2 + ab)} \\ & \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{ax(a^2 + bc) + by(b^2 + ca) + cz(c^2 + ab)} \\ & \stackrel{\text{CBS}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2} \cdot \sqrt{(a^2 + bc)^2 + (b^2 + ca)^2 + (c^2 + ab)^2}} \stackrel{?}{\geq} \frac{\sqrt{3}}{2} \frac{a^2 + b^2 + c^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \\ & \Leftrightarrow 4(a^2 + b^2 + c^2)^2 \stackrel{?}{\geq} 3 \left( (a^2 + bc)^2 + (b^2 + ca)^2 + (c^2 + ab)^2 \right) \\ & \Leftrightarrow 4 \sum_{\text{cyc}} a^4 + 8 \sum_{\text{cyc}} a^2b^2 \stackrel{?}{\geq} 3 \sum_{\text{cyc}} a^4 + 3 \sum_{\text{cyc}} a^2b^2 + 6abc \sum_{\text{cyc}} a \\ & \Leftrightarrow \sum_{\text{cyc}} a^4 + 5 \sum_{\text{cyc}} a^2b^2 \stackrel{?}{\geq} 6abc \sum_{\text{cyc}} a \rightarrow \text{true} \because \sum_{\text{cyc}} a^4 + 5 \sum_{\text{cyc}} a^2b^2 \geq 6 \sum_{\text{cyc}} a^2b^2 \\ & \geq 6abc \sum_{\text{cyc}} a \because \frac{a^3}{x(a^2 + bc)} + \frac{b^3}{y(b^2 + ca)} + \frac{c^3}{z(c^2 + ab)} \geq \frac{\sqrt{3}}{2} \frac{a^2 + b^2 + c^2}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \\ & \quad \forall a, b, c, x, y, z > 0, \text{''} = \text{''} \text{ iff } (a = b = c \text{ and } x = y = z) \text{ (QED)} \end{aligned}$$