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If $a, b, c \geq 0$ with $ab + bc + ca \neq 0$, then prove that :

$$\sum_{cyc} \frac{a^3}{b^2 - bc + c^2} \geq a + b + c + AB \sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right),$$

$$\text{where } A = \frac{2(a^3 + b^3 + c^3)}{2(a^2 + b^2 + c^2) + (a-b)^2 + (b-c)^2 + (c-a)^2};$$

$$B = \frac{(a+b)(b+c)(c+a)}{(a^3 + b^3)(b^3 + c^3)(c^3 + a^3)}$$

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Let $a^2 - ab + b^2 = x, b^2 - bc + c^2 = y, c^2 - ca + a^2 = z$ and then :

$$(c^2 - ca + a^2) - (b^2 - bc + c^2) = (a-b)(a+b) - c(a-b)$$

$$= (a-b)(a+b-c) \Rightarrow (z-y)^2 = (a-b)^2(a+b-c)^2 \text{ and analogs}$$

$$\therefore B \sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right)$$

$$= \frac{(a+b)(b+c)(c+a)\left((z-y)^2 x + (z-x)^2 y + (x-y)^2 z\right)}{(a+b)(a^2 - ab + b^2)(b+c)(b^2 - bc + c^2)(c+a)(c^2 - ca + a^2)}$$

$$= \frac{1}{xyz} \cdot \left(\sum_{cyc} \left(xy\left(\sum_{cyc} x - z\right)\right) - 6xyz\right)$$

$$= \frac{\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz}{xyz} \therefore AB \sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right)$$

$$= \frac{\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \cdot \left(\frac{\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)}{xyz} - 9\right)$$

$$= \frac{\sum_{cyc} a^3}{\sum_{cyc} x} \cdot \frac{\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)}{xyz} - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab}$$

$$\therefore AB \sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right)$$

$$= \frac{\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} xy\right)}{xyz} - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \rightarrow (1)$$

$$\text{Again, } \sum_{cyc} \frac{a^3}{b^2 - bc + c^2} = \frac{a^3 zx + b^3 xy + c^3 yz}{xyz} \rightarrow (2) \therefore (1), (2)$$

$$\Rightarrow \sum_{cyc} \frac{a^3}{b^2 - bc + c^2} \geq a + b + c + AB \sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right)$$

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$$\Leftrightarrow \frac{\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} xy\right) - \left(a^3zx + b^3xy + c^3yz\right)}{xyz} + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0$$

$$\Leftrightarrow \frac{a^3xy + a^3yz + b^3yz + b^3zx + c^3xy + c^3zx}{xyz} + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0$$

$$\Leftrightarrow \frac{xy(c+a)z + yz(a+b)x + zx(b+c)y}{xyz} + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0$$

$$\Leftrightarrow 2\sum_{cyc} a + \sum_{cyc} a - \frac{9\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \le 0 \Leftrightarrow \frac{3\sum_{cyc} a^3}{2\sum_{cyc} a^2 - \sum_{cyc} ab} \ge \sum_{cyc} a$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \ge \sum_{cyc} a^2b + \sum_{cyc} ab^2 \rightarrow \text{true via Schur}$$

$$\therefore \sum_{cyc} \frac{a^3}{b^2 - bc + c^2} \ge a + b + c + AB\sum_{cyc} (a-b)^2(a+b-c)^2\left((a-b)^2 + ab\right)$$

$\forall\, a, b, c \ge 0$ with $ab + bc + ca \ne 0$, " = " iff $(a = b = c)$ or $(a = 0, b = c \ne 0)$ or $(b = 0, c = a \ne 0)$ or $(c = 0, a = b \ne 0)$ (QED)