ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{b(b+c)^{3n}+a(a+c)^{3n}}{c(a+b)}+\frac{b(a+b)^{3n}+c(a+c)^{3n}}{a(b+c)}+\frac{a(a+b)^{3n}+c(b+c)^{3n}}{b(a+c)}\geq 3.8^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{b(b+c)^{3n}+a(a+c)^{3n}}{c(a+b)}+\frac{b(a+b)^{3n}+c(a+c)^{3n}}{a(b+c)}+\frac{a(a+b)^{3n}+c(b+c)^{3n}}{b(a+c)}=\sum_{cyc}\left((b+c)^{3n}\left(\frac{b}{c(a+b)}+\frac{c}{b(c+a)}\right)\right)$$

$$=\sum_{cyc}\left(\frac{(b+c)^{3n}}{(a+b)(c+a)}.\frac{b^2c+b^2a+c^2a+c^2b}{bc}\right)$$

$$\geq\sum_{cyc}\left(\frac{(b+c)^{3n}}{(a+b)(c+a)}.\frac{bc(b+c)+\frac{a(b+c)^2}{2}}{bc}\right)$$

$$=\sum_{cyc}\left(\frac{(b+c)^{3n}}{2}.\frac{(2bc+ab+ac)(b+c)}{(a+b)(c+a)bc}\right)$$

$$\stackrel{A-G}{\geq}3.\sqrt[3]{\frac{(a+b)^{3n}.(b+c)^{3n}.(c+a)^{3n}}{8}.\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{a^2b^2c^2(a+b)(b+c)(c+a)}}$$

$$\stackrel{abc=1}{=}3\left(\frac{(a+b)(b+c)}{(c+a)}\right)^n.\sqrt[3]{\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{8abc(a+b)(b+c)(c+a)}}$$

$$\stackrel{\text{Cesaro and }\because n\geq 1}{\geq}3(8abc)^n.\sqrt[3]{\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{8abc(a+b)(b+c)(c+a)}}$$

$$\stackrel{abc=1}{=}3.8^n.\sqrt[3]{\frac{(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)}{8abc(a+b)(b+c)(c+a)}}\stackrel{?}{\geq}3.8^n$$

$$\Leftrightarrow(2bc+ab+ca)(2ca+ab+bc)(2ab+bc+ca)\stackrel{?}{\geq}8abc(a+b)(b+c)(c+a)$$

$$\Leftrightarrow 2\sum_{cyc}a^3b^3\geq abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$\Leftrightarrow 2\sum_{cyc}a^3b^3+3a^2b^2c^2\geq abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+3a^2b^2c^2\rightarrow\text{true}$$

$$\because \sum_{cyc} a^3b^3 + 3a^2b^2c^2 \overset{Schur}{\geq} abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \ and \ \sum_{cyc} a^3b^3 \overset{A-G}{\geq} 3a^2b^2c^2$$

$$\therefore \frac{b(b+c)^{3n} + a(a+c)^{3n}}{c(a+b)} + \frac{b(a+b)^{3n} + c(a+c)^{3n}}{a(b+c)} + \frac{a(a+b)^{3n} + c(b+c)^{3n}}{b(a+c)}$$

$$\geq 3.8^n \ \forall a,b,c > 0, '' ='' \ \text{iff } a = b = c = 1 \ (QED)$$