ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ such that $abc = 1$, then prove that:

$$\frac{a^5b^5(a^2+b^2)}{c^5(a^6+b^6)}+\frac{b^5c^5(b^2+c^2)}{a^5(b^6+c^6)}+\frac{c^5a^5(c^2+a^2)}{b^5(c^6+a^6)}\geq 3$$

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We shall first prove that $\forall\,\Delta ABC$,

$$\boxed{(s^2-8Rr-2r^2)^6\geq 27r^5s^4(4R-5r)^3\to(\bullet)}\ \text{and}$$

$$\because\ 10240R^3-26880R^2r+23520Rr^2-6860r^3$$
$$=(R-2r)(10240R^2-6400Rr+10720r^2)+14580r^3\overset{\text{Euler}}{\geq}14580r^3$$
$$>0\ \text{and}\ 61440R^4-216768R^3r+288720R^2r^2-172740Rr^3+39390r^4$$
$$=(R-2r)(61440R^3-93888R^2r+100944Rr^2+29148r^3)+97686r^4$$
$$\overset{\text{Euler}}{\geq}97686r^4>0\ \therefore\ \text{via Gerretsen,}$$

$$\boxed{\boxed{P}}=(s^2-16Rr+5r^2)^6+(48R-42r^2)(s^2-16Rr+5r^2)^5$$
$$+r^2(960R^2-1680Rr+735r^2)(s^2-16Rr+5r^2)^4$$
$$+r^3(10240R^3-26880R^2r+23520Rr^2-6860r^3)(s^2-16Rr+5r^2)^3$$
$$+r^4\begin{pmatrix}61440R^4-216768R^3r+288720R^2r^2\\-172740Rr^3+39390r^4\end{pmatrix}(s^2-16Rr+5r^2)^2\geq 0$$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove:

$$(s^2-8Rr-2r^2)^6-27r^5s^4(4R-5r)^3\geq P$$

$$\Leftrightarrow\begin{pmatrix}49152R^5-228864R^4r+432480R^3r^2-410280R^2r^3\\+191310Rr^4-33648r^5\end{pmatrix}s^2$$
$$\boxed{\overset{(*)}{\geq}}\ r\begin{pmatrix}720896R^6-3452928R^5r+6827520R^4r^2-7060400R^3r^3\\+3955620R^2r^4-1107609Rr^5+117734r^6\end{pmatrix}$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\geq}\begin{pmatrix}49152R^5-228864R^4r+432480R^3r^2\\-410280R^2r^3+191310Rr^4-33648r^5\end{pmatrix}(16Rr-5r^2)$$
$$\overset{?}{\geq}r\begin{pmatrix}720896R^6-3452928R^5r+6827520R^4r^2-7060400R^3r^3\\+3955620R^2r^4-1107609Rr^5+117734r^6\end{pmatrix}$$
$$\Leftrightarrow 65536t^6-454656t^5+1236480t^4-1666480t^3+1156740t^2$$
$$-387309t+50506\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\boxed{(t-2)\begin{pmatrix}(t-2)\big((t-2)(65536t^3-61440t^2+81408t+83536)+189540\big)\\+19683\end{pmatrix}\overset{?}{\geq}0}$$

$$\to\text{true}\ \because\ t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\Rightarrow(\bullet)\text{ is true}$$

$$\text{Now,}\ \frac{a^5b^5(a^2+b^2)}{c^5(a^6+b^6)}+\frac{b^5c^5(b^2+c^2)}{a^5(b^6+c^6)}+\frac{c^5a^5(c^2+a^2)}{b^5(c^6+a^6)}\overset{abc=1}{=}$$
$$\sum_{\text{cyc}}\frac{b^8c^8(b^2+c^2)}{a^2(b^2+c^2)(b^4-b^2c^2+c^4)}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{\text{cyc}}b^4c^4\right)^2}{\left(\sum_{\text{cyc}}a^2b^2\right)\left(\sum_{\text{cyc}}a^2\right)-6a^2b^2c^2}$$

$$= \frac{\left(\sum_{cyc} x^2y^2\right)^2}{\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 6xyz}\ (x = a^2, y = b^2, z = c^2) \overset{?}{\geq} 3 \overset{xyz=1}{=} 3(xyz)^{\frac{5}{3}}$$

$$\Leftrightarrow \boxed{\left(\sum_{cyc} x^2y^2\right)^6 \underset{(**)}{\overset{?}{\geq}} 27(xyz)^5.\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 6xyz\right)^3}$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \rightarrow$ (i)

$\Rightarrow x = s - X, y = s - Y, z = s - Z$ and such substitutions $\Rightarrow$
$\sum_{cyc} xy = \sum_{cyc}(s - X)(s - Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \rightarrow$ (ii) and

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (i) and (ii)}}{=}$$

$$(4Rr + r^2)^2 - 2\left(\prod_{cyc}(s - X)\right).s$$

$$= (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 = r^2\left((4R + r)^2 - 2s^2\right) \rightarrow \text{(iii)}$$

$\therefore$ via (i), (ii) and (iii), $(**) \Leftrightarrow$

$$r^{12}\left((4R + r)^2 - 2s^2\right)^6 \geq 27r^{10}s^5\left(s(4Rr + r^2) - 6r^2s\right)^3$$

$$\Leftrightarrow \left((4R + r)^2 - 2s^2\right)^6 \geq 27rs^2.\left(4R + r - \frac{6rs^2}{s^2}\right)^3.s^6$$

$$\Leftrightarrow \left(\sum_{cyc} r_a^2\right)^6 \geq 27r_ar_br_c.\left(\sum_{cyc} r_a - \frac{6r_ar_br_c}{\sum_{cyc} r_ar_b}\right)^3.\left(\sum_{cyc} r_ar_b\right)^3$$

$$\Leftrightarrow \left(\sum_{cyc} r_a^2\right)^6 \geq 27r_ar_br_c\left(\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} r_ar_b\right) - 6r_ar_br_c\right)^3$$

$$\Leftrightarrow \boxed{\left(\sum_{cyc} x'^2\right)^6 \overset{(***)}{\geq} 27x'y'z'\left(\left(\sum_{cyc} x'\right)\left(\sum_{cyc} x'y'\right) - 6x'y'z'\right)^3}\ \left(\begin{matrix} x' = r_a, \\ y' = r_b, z' = r_c\end{matrix}\right)$$

Assigning $y' + z' = X', z' + x' = Y', x' + y' = Z' \Rightarrow X' + Y' - Z' = 2z' > 0, Y' + Z' - X' = 2x' > 0$ and $Z' + X' - Y' = 2y' > 0 \Rightarrow X' + Y' > Z'$, $Y' + Z' > X', Z' + X' > Y' \Rightarrow X', Y', Z'$ form sides of a triangle with semiperimeter, circumradius and inradius $= s', R', r'$ (say) yielding

$$2\sum_{cyc} x' = \sum_{cyc} X' = 2s' \Rightarrow \sum_{cyc} x' = s' \rightarrow (1)$$

$\Rightarrow x' = s' - X', y' = s' - Y', z' = s' - Z'$ and such substitutions

$$\Rightarrow \sum_{cyc} x'y' = \sum_{cyc} (s' - X')(s' - Y') \Rightarrow \sum_{cyc} x'y' = 4R'r' + r'^2 \rightarrow (2) \text{ and}$$

$$\sum_{cyc} x'^2 = \left(\sum_{cyc} x'\right)^2 - 2\sum_{cyc} x'y' \overset{\text{via } (1) \text{ and } (2)}{=} s'^2 - 2(4R'r' + r'^2)$$

$$\Rightarrow \sum_{cyc} x'^2 = s'^2 - 8R'r' - 2r'^2 \rightarrow (3) \therefore \text{via } (1), (2) \text{ and } (3), (***)$$

$$\Leftrightarrow \left(s'^2 - 8R'r' - 2r'^2\right)^6 \geq 27r'^2 s'\left(s'\left(4R'r' + r'^2\right) - 6r'^2 s'\right)^3$$

$$\Leftrightarrow \boxed{\left(s'^2 - 8R'r' - 2r'^2\right)^6 \geq 27r'^5 s'^4 (4R' - 5r')^3} \rightarrow \text{true via } (\bullet)$$

$$\Rightarrow (***) \Rightarrow (**) \text{ is true} \therefore \frac{a^5b^5(a^2 + b^2)}{c^5(a^6 + b^6)} + \frac{b^5c^5(b^2 + c^2)}{a^5(b^6 + c^6)} + \frac{c^5a^5(c^2 + a^2)}{b^5(c^6 + a^6)} \geq 3$$

$$\forall\, a, b, c > 0 \,\middle|\, abc = 1, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$