ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that $\forall\, n \in \mathbb{N}$, we have :

$$\frac{b(b^2+bc+c^2)^{3n}+a(a^2+ac+c^2)^{3n}}{c(a+b)}+\frac{b(a^2+ab+b^2)^{3n}+c(a^2+ac+c^2)^{3n}}{a(b+c)}$$
$$+\frac{a(a^2+ab+b^2)^{3n}+c(b^2+bc+c^2)^{3n}}{b(a+c)}\ge 3^{3n+1}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b(a^2+ab+b^2)^{3n}+c(a^2+ac+c^2)^{3n}}{a(b+c)}\overset{A-G}{\ge}\sum_{cyc}\frac{b(3ab)^{3n}+c(3ac)^{3n}}{a(b+c)}$$
$$=\sum_{cyc}\left(\frac{3^{3n}.a^{3n-1}}{b+c}.(b.b^{3n}+c.c^{3n})\right)\overset{Chebyshev}{\ge}\sum_{cyc}\left(\frac{3^{3n}.a^{3n-1}}{2(b+c)}.(b+c)(b^{3n}+c^{3n})\right)$$
$$=\frac{3^{3n}}{2}.\sum_{cyc}a^{3n-1}.(b^{3n}+c^{3n})\overset{A-G}{\ge}\frac{3.3^{3n}}{2}.\sqrt[3]{(abc)^{3n-1}.\prod_{cyc}(b^{3n}+c^{3n})}$$
$$\overset{Cesaro}{\ge}\frac{3.3^{3n}}{2}.\sqrt[3]{(abc)^{3n-1}.8(abc)^{3n}}\overset{abc=1}{=}\frac{3.3^{3n}.2}{2}=3^{3n+1}$$
$$\therefore\frac{b(b^2+bc+c^2)^{3n}+a(a^2+ac+c^2)^{3n}}{c(a+b)}+\frac{b(a^2+ab+b^2)^{3n}+c(a^2+ac+c^2)^{3n}}{a(b+c)}$$
$$+\frac{a(a^2+ab+b^2)^{3n}+c(b^2+bc+c^2)^{3n}}{b(a+c)}\ge 3^{3n+1}\ \forall\, a,b,c>0\,\middle|\,abc=1$$

and $\forall\, n \in \mathbb{N}$, " = " iff $a = b = c = 1$ (QED)