ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c}\ge 3$$

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$$\frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c}$$

$$=\left(\frac{c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}}{b+c}\right)+\left(\frac{b^{2022}c^{2026}}{a+c}+\frac{(bc)^{2024}}{a+b}\right)+\left(\frac{a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}}{a+c}\right)$$

$$=c^{2022}a^{2024}.\left(\frac{a^2}{a+b}+\frac{c^2}{b+c}\right)+b^{2022}c^{2024}.\left(\frac{c^2}{c+a}+\frac{b^2}{a+b}\right)$$

$$+a^{2022}b^{2024}.\left(\frac{b^2}{b+c}+\frac{a^2}{c+a}\right)$$

$$\overset{\text{Bergstrom}}{\ge} c^{2022}a^{2024}.\frac{(c+a)^2}{(a+b)+(b+c)}+b^{2022}c^{2024}.\frac{(b+c)^2}{(c+a)+(a+b)}$$

$$+a^{2022}b^{2024}.\frac{(a+b)^2}{(b+c)+(c+a)}$$

$$\overset{A-G}{\ge} 3.\sqrt[3]{(abc)^{4046}.\frac{(a+b)^2}{(b+c)+(c+a)}.\frac{(b+c)^2}{(c+a)+(a+b)}.\frac{(c+a)^2}{(a+b)+(b+c)}}$$

$$\overset{abc=1}{=} 3.\sqrt[3]{\frac{1}{abc}.\frac{(a+b)^2}{(b+c)+(c+a)}.\frac{(b+c)^2}{(c+a)+(a+b)}.\frac{(c+a)^2}{(a+b)+(b+c)}}$$

$$\therefore \frac{(bc)^{2024}+c^{2022}a^{2026}}{a+b}+\frac{(ca)^{2024}+a^{2022}b^{2026}}{b+c}+\frac{(ab)^{2024}+b^{2022}c^{2026}}{a+c}$$

$$\ge 3.\sqrt[3]{\frac{1}{abc}.\frac{(a+b)^2}{(b+c)+(c+a)}.\frac{(b+c)^2}{(c+a)+(a+b)}.\frac{(c+a)^2}{(a+b)+(b+c)}}\to(1)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s \Rightarrow \sum_{cyc}a=s \Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=(s-x)(s-y)(s-z) \Rightarrow abc=r^2s$ and via such substitutions,

$$\text{RHS of }(1)\ge 3.\sqrt[3]{\frac{1}{r^2s}.\frac{z^2}{x+y}.\frac{x^2}{y+z}.\frac{y^2}{z+x}}\overset{?}{\ge}3 \Leftrightarrow \frac{16R^2r^2s^2}{r^2s.2s(s^2+2Rr+r^2)}\overset{?}{\ge}1$$

$$\Leftrightarrow s^2\overset{?}{\le}8R^2-2Rr-r^2 \Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r)\overset{?}{\le}0\to\text{true}$$

$$\because\ s^2 - 4R^2 - 4Rr - 3r^2 \overset{\text{Gerretsen}}{\leq} 0 \ and\ -2(2R + r)(R - 2r) \overset{\text{Euler}}{\leq} 0$$

$$\Rightarrow \text{RHS of } (1) \geq 3 \overset{\text{via } (1)}{\Rightarrow}$$

$$\frac{(bc)^{2024} + c^{2022}a^{2026}}{a + b} + \frac{(ca)^{2024} + a^{2022}b^{2026}}{b + c} + \frac{(ab)^{2024} + b^{2022}c^{2026}}{a + c} \geq 3$$

$$\forall\, a, b, c > 0 \,\middle|\, abc = 1, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$