ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a^7c^2+b^9}{ab(a^3+b^3)}+\frac{b^7a^2+c^9}{bc(b^3+c^3)}+\frac{c^7b^2+a^9}{ac(a^3+c^3)}\geq 3$$

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$$\text{Let } a=\frac{1}{x}, b=\frac{1}{y}, c=\frac{1}{z} \text{ and then : } \frac{a^7c^2}{ab(a^3+b^3)} \overset{\frac{1}{ab}=xy}{=} \frac{\frac{1}{x^7z^2}.xy}{\frac{1}{x^3}+\frac{1}{y^3}}$$

$$=\frac{\frac{1}{x^4}\cdot\frac{y}{x^2z^2}}{\frac{1}{x^3}+\frac{1}{y^3}} \overset{y=\frac{1}{zx}}{=} \frac{\frac{1}{x^4}\cdot\frac{1}{x^3z^3}}{\frac{x^3+y^3}{x^3y^3}}=\frac{\frac{1}{x^4}\cdot\frac{y^3}{z^3}}{x^3+y^3}=\frac{\frac{1}{x^4}\cdot\left(\frac{y}{z}\right)^4}{x^3\cdot\frac{y}{z}+y^3\cdot\frac{y}{z}} \overset{\frac{1}{zx}=y}{=} \frac{y^4.y^4}{x^3y.xy+y^4.xy}$$

$$=\frac{y^8}{xy^2(x^3+y^3)} \Rightarrow \frac{a^7c^2}{ab(a^3+b^3)}=\frac{y^6}{x(x^3+y^3)} \text{ and analogously,}$$

$$\frac{b^7a^2}{bc(b^3+c^3)}=\frac{z^6}{y(y^3+z^3)} \text{ and } \frac{c^7b^2}{ac(a^3+c^3)}=\frac{x^6}{z(z^3+x^3)}$$

$$\therefore \frac{a^7c^2}{ab(a^3+b^3)}+\frac{b^7a^2}{bc(b^3+c^3)}+\frac{c^7b^2}{ac(a^3+c^3)}$$

$$=\frac{y^6}{x(x^3+y^3)}+\frac{z^6}{y(y^3+z^3)}+\frac{x^6}{z(z^3+x^3)} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}x^3\right)^2}{\sum_{cyc}\left(x\left(\sum_{cyc}x^3-z^3\right)\right)}$$

$$=\frac{\left(\sum_{cyc}x^3\right)^2}{\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^3\right)-\sum_{cyc}x^3y} \geq \frac{\left(\sum_{cyc}x^3\right)^2}{\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^3\right)-3}$$

$$\left(\because \sum_{cyc}x^3y \overset{A-G}{\geq} 3.\sqrt[3]{(xyz)^4} \overset{xyz=1}{=} 3\right) \geq \frac{\left(\sum_{cyc}x^3\right)^2}{\sqrt[3]{9\sum_{cyc}x^3}.\left(\sum_{cyc}x^3\right)-3}$$

$$\left(\because \sum_{cyc}x^3 \overset{\text{Holder}}{\geq} \frac{1}{9}\left(\sum_{cyc}x\right)^3 \Rightarrow \sum_{cyc}x \leq \sqrt[3]{9\sum_{cyc}x^3}\right)=\frac{t^2}{t.\sqrt[3]{9t}-3}\left(t=\sum_{cyc}x^3\right) \overset{?}{\geq} \frac{3}{2}$$

$$\Leftrightarrow 2t^2+9 \overset{?}{\geq} 3t.\sqrt[3]{9t} \Leftrightarrow \left(2t^2+9\right)^3 \overset{?}{\geq} 243t^4 \Leftrightarrow 8t^6-135t^4+486t^2+729 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-3)^2\left(8t^4+48t^3+81t^2+54t+81\right) \overset{?}{\geq} 0 \rightarrow \text{true,}$$

$$''='' \text{ iff } t=\sum_{cyc}x^3=3 \text{ and } \because \sum_{cyc}x^3 \overset{A-G}{\geq} 3xyz \overset{xyz=1}{=} 3 \therefore ''='' \text{ iff } x=y=z=1$$

$$\therefore \frac{a^7c^2}{ab(a^3+b^3)}+\frac{b^7a^2}{bc(b^3+c^3)}+\frac{c^7b^2}{ac(a^3+c^3)} \geq \frac{3}{2}, ''='' \text{ iff } a=b=c=1 \rightarrow (1)$$

$$\text{Now, } a^4+b^4+b^4+b^4 \overset{A-G}{\geq} 4ab^3, b^4+c^4+c^4+c^4 \overset{A-G}{\geq} 4bc^3,$$

$$c^4 + a^4 + a^4 + a^4 \overset{A-G}{\geq} 4ca^3 \therefore 4\sum_{cyc} a^4 \geq 4\sum_{cyc} ab^3 \Rightarrow \sum_{cyc} ab^3 \leq \sum_{cyc} a^4 \to (i)$$

$$\text{We have}: \frac{b^9}{ab(a^3+b^3)} + \frac{c^9}{bc(b^3+c^3)} + \frac{a^9}{ac(a^3+c^3)}$$

$$= \frac{b^8}{a^4+ab^3} + \frac{c^8}{b^4+bc^3} + \frac{a^8}{c^4+ca^3} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^4\right)^2}{\sum_{cyc} a^4 + \sum_{cyc} ab^3} \overset{\text{via (i)}}{\geq} \frac{\left(\sum_{cyc} a^4\right)^2}{2\sum_{cyc} a^4}$$

$$= \frac{1}{2}\sum_{cyc} a^4 \overset{A-G}{\geq} \frac{3}{2}.\sqrt[3]{(abc)^4} \overset{abc=1}{=} \frac{3}{2}$$

$$\Rightarrow \frac{b^9}{ab(a^3+b^3)} + \frac{c^9}{bc(b^3+c^3)} + \frac{a^9}{ac(a^3+c^3)} \geq \frac{3}{2}, '' ='' \text{ iff } a = b = c = 1 \to (2)$$

$$\therefore (1) + (2) \Rightarrow \frac{a^7c^2 + b^9}{ab(a^3+b^3)} + \frac{b^7a^2 + c^9}{bc(b^3+c^3)} + \frac{c^7b^2 + a^9}{ac(a^3+c^3)} \geq 3,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$