ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a^2b^2(b^7c^2+a^9)}{c(a^5+b^5)}+\frac{b^2c^2(c^7a^2+b^9)}{a(b^5+c^5)}+\frac{c^2a^2(a^7b^2+c^9)}{b(a^5+c^5)}\geq 3$$

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Solution by Soumava Chakraborty-Kolkata-India

$$3b^5+2a^5\overset{A-G}{\geq}5b^3a^2,\ 3c^5+2b^5\overset{A-G}{\geq}5c^3b^2,\ 3a^5+2c^5\overset{A-G}{\geq}5a^3c^2$$

$$\therefore \text{ via summation, } 5\sum_{cyc}a^5\geq 5\sum_{cyc}a^2b^3\Rightarrow\sum_{cyc}a^2b^3\leq\sum_{cyc}a^5\to(1)$$

$$\text{Now, }\frac{a^2b^2(b^7c^2+a^9)}{c(a^5+b^5)}+\frac{b^2c^2(c^7a^2+b^9)}{a(b^5+c^5)}+\frac{c^2a^2(a^7b^2+c^9)}{b(a^5+c^5)}\overset{abc=1}{=}$$

$$\frac{b^7}{c(a^5+b^5)}+\frac{a^9}{c^3(a^5+b^5)}+\frac{c^7}{a(b^5+c^5)}+\frac{b^9}{a^3(b^5+c^5)}+\frac{a^7}{b(a^5+c^5)}+\frac{c^9}{b^3(a^5+c^5)}$$

$$=\frac{b^{10}}{b^3c(a^5+b^5)}+\frac{a^{10}}{c^3a(a^5+b^5)}+\frac{c^{10}}{c^3a(b^5+c^5)}+\frac{b^{10}}{a^3b(b^5+c^5)}+\frac{a^{10}}{a^3b(a^5+c^5)}+\frac{c^{10}}{b^3c(a^5+c^5)}$$

$$=a^{10}.\left(\frac{1}{c^3a(a^5+b^5)}+\frac{1}{a^3b(a^5+c^5)}\right)+b^{10}.\left(\frac{1}{b^3c(a^5+b^5)}+\frac{1}{a^3b(b^5+c^5)}\right)$$

$$+c^{10}.\left(\frac{1}{c^3a(b^5+c^5)}+\frac{1}{b^3c(a^5+c^5)}\right)\overset{\substack{A-G\\ \text{and}\\ \because abc=1}}{\geq}$$

$$\frac{2a^{10}}{\sqrt{a^3c^2(c^5+a^5)(a^5+b^5)}}+\frac{2b^{10}}{\sqrt{b^3a^2(a^5+b^5)(b^5+c^5)}}+\frac{2c^{10}}{\sqrt{c^3b^2(b^5+c^5)(c^5+a^5)}}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{2\left(\sum_{cyc}a^5\right)^2}{\sum_{cyc}\sqrt{a^2b^3(a^5+b^5)(b^5+c^5)}}\overset{CBS}{\geq}\frac{2\left(\sum_{cyc}a^5\right)^2}{\sqrt{\sum_{cyc}a^2b^3}.\sum_{cyc}\sqrt{(a^5+b^5)(b^5+c^5)}}$$

$$\overset{\substack{\text{via }(1)\\ \text{and}\\ \because 3\sum_{cyc}xy\leq\left(\sum_{cyc}x\right)^2}}{\geq}\frac{2\left(\sum_{cyc}a^5\right)^2}{\sqrt{\sum_{cyc}a^5}.\sum_{cyc}\sqrt{\frac{\left(\sum_{cyc}(b^5+c^5)\right)^2}{3}}}=\frac{2\sqrt{3}.\left(\sum_{cyc}a^5\right)^2}{\sqrt{\sum_{cyc}a^5}.\left(2\sum_{cyc}a^5\right)}$$

$$=\sqrt{3\sum_{cyc}a^5}\overset{A-G}{\geq}\sqrt{9(abc)^{\frac{5}{3}}}\overset{abc=1}{=}3$$

$$\therefore\frac{a^2b^2(b^7c^2+a^9)}{c(a^5+b^5)}+\frac{b^2c^2(c^7a^2+b^9)}{a(b^5+c^5)}+\frac{c^2a^2(a^7b^2+c^9)}{b(a^5+c^5)}\geq 3$$

$$\forall\, a,b,c>0\,\big|\,abc=1,\text{''}=\text{''}\text{ iff }a=b=c=1\text{ (QED)}$$