ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^3+b^3+c^3=3$, then prove that:

$$\frac{1}{a^4+b^5+c^6}+\frac{1}{b^4+c^5+a^6}+\frac{1}{c^4+a^5+b^6}\le 1$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via weighted AM}-\text{HM inequality}, \frac{a.a^3+b^2.b^3+c^3.c^3}{a^3+b^3+c^3}$$

$$\ge\frac{a^3+b^3+c^3}{\frac{a^3}{a}+\frac{b^3}{b^2}+\frac{c^3}{c^3}}\overset{\because a^3+b^3+c^3=3}{\Rightarrow} a^4+b^5+c^6\ge\frac{9}{a^2+b+1}$$

$$\Rightarrow\frac{1}{a^4+b^5+c^6}\le\frac{a^2+b+1}{9}\text{ and analogs}$$

$$\therefore\frac{1}{a^4+b^5+c^6}+\frac{1}{b^4+c^5+a^6}+\frac{1}{c^4+a^5+b^6}\le\sum_{cyc}\frac{a^2+b+1}{9}$$

$$\le\frac{1}{9}\left(\sum_{cyc}a^2+\sum_{cyc}a+3\right)\overset{?}{\le}1\Leftrightarrow\sum_{cyc}a^2+\sum_{cyc}a\underset{(*)}{\overset{?}{\le}}6$$

$$\text{Now, via Power Mean Inequality}, \left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}}\ge\left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}$$

$$\overset{\because a^3+b^3+c^3=3}{\Rightarrow}1\ge\sqrt{\frac{\sum_{cyc}a^2}{3}}\Rightarrow\sum_{cyc}a^2\le 3\to(1)$$

$$\text{Again, via Holder}, a^3+b^3+c^3\ge\frac{1}{9}\left(\sum_{cyc}a\right)^3\overset{\because a^3+b^3+c^3=3}{\Rightarrow}3\ge\frac{1}{9}\left(\sum_{cyc}a\right)^3$$

$$\Rightarrow\sum_{cyc}a\le 3\to(2)\therefore(1)+(2)\Rightarrow\text{LHS of }(*)\le 3+3\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{1}{a^4+b^5+c^6}+\frac{1}{b^4+c^5+a^6}+\frac{1}{c^4+a^5+b^6}\le 1\ \forall\, a,b,c>0\,\big|\,a^3+b^3+c^3=3,$$

$$''=''\text{ iff } a=b=c=1\ (\text{QED})$$