ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then $\forall\, m,n\in\mathbb{N}$, prove that :

$$\frac{b^{2m}.\sqrt{b^n+c^n}+a^{2m}.\sqrt{a^n+c^n}}{a^m+b^m}+\frac{b^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{a^n+c^n}}{b^m+c^m}$$
$$+\frac{a^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{b^n+c^n}}{a^m+c^m}\ge 3\sqrt{2}$$

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$$\text{LHS}=\sqrt{a^n+b^n}.\left(\frac{b^{2m}}{b^m+c^m}+\frac{a^{2m}}{c^m+a^m}\right)$$
$$+\sqrt{b^n+c^n}.\left(\frac{b^{2m}}{a^m+b^m}+\frac{c^{2m}}{c^m+a^m}\right)+\sqrt{c^n+a^n}.\left(\frac{a^{2m}}{a^m+b^m}+\frac{c^{2m}}{b^m+c^m}\right)$$
$$\overset{\text{Bergstrom}}{\ge}\sqrt{a^n+b^n}.\frac{(a^m+b^m)^2}{\left((b^m+c^m)+(c^m+a^m)\right)}$$
$$+\sqrt{b^n+c^n}.\frac{(b^m+c^m)^2}{\left((c^m+a^m)+(a^m+b^m)\right)}+\sqrt{c^n+a^n}.\frac{(c^m+a^m)^2}{\left((a^m+b^m)+(b^m+c^m)\right)}\overset{A-G}{\ge}$$
$$3.\sqrt[3]{\sqrt{\prod_{\text{cyc}}(b^n+c^n)}.\frac{(a^m+b^m)^2.(b^m+c^m)^2.(c^m+a^m)^2}{\prod_{\text{cyc}}\left((b^m+c^m)+(c^m+a^m)\right)}}\overset{\text{Cesaro}}{\ge}$$
$$3.\sqrt[3]{\sqrt{8a^nb^nc^n}.\frac{(a^m+b^m)^2.(b^m+c^m)^2.(c^m+a^m)^2}{\left((b^m+c^m)+(c^m+a^m)\right)\left((c^m+a^m)+(a^m+b^m)\right)\left((a^m+b^m)+(b^m+c^m)\right)}}$$
$$\overset{abc=1}{=}3\sqrt{2}.\sqrt[3]{\frac{(a^m+b^m)^2.(b^m+c^m)^2.(c^m+a^m)^2}{a^mb^mc^m.\prod_{\text{cyc}}\left((b^m+c^m)+(c^m+a^m)\right)}}\overset{?}{\ge}3\sqrt{2}$$
$$\Leftrightarrow\sqrt[3]{\frac{1}{xyz}.\frac{(x+y)^2.(y+z)^2.(z+x)^2}{\left((y+z)+(z+x)\right)\left((z+x)+(x+y)\right)\left((x+y)+(y+z)\right)}}\overset{?}{\ge}1\to(1)$$
$$(x=a^m,y=b^m,z=c^m)$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

yielding $2\sum_{\text{cyc}}x=\sum_{\text{cyc}}X=2s\Rightarrow\sum_{\text{cyc}}x=s\Rightarrow x=s-X, y=s-Y, z=s-Z$ and

such substitutions $\Rightarrow xyz = (s-X)(s-Y)(s-Z) \Rightarrow xyz = r^2s \therefore$ LHS of (1)

$$\geq \sqrt[3]{\frac{1}{r^2s}\cdot\frac{Z^2X^2Y^2}{(X+Y)(Y+Z)(Z+X)}} \overset{?}{\geq} 1 \Leftrightarrow \frac{16R^2r^2s^2}{r^2s.\,2s(s^2+2Rr+r^2)} \overset{?}{\geq} 1$$

$$\Leftrightarrow s^2 \overset{?}{\leq} 8R^2-2Rr-r^2 \Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r) \overset{?}{\leq} 0 \rightarrow \text{true}$$

$$\because s^2-4R^2-4Rr-3r^2 \overset{\text{Gerretsen}}{\leq} 0 \text{ and } -2(2R+r)(R-2r) \overset{\text{Euler}}{\leq} 0 \Rightarrow (1) \text{ is true}$$

$$\Rightarrow \frac{b^{2m}.\sqrt{b^n+c^n}+a^{2m}.\sqrt{a^n+c^n}}{a^m+b^m}+\frac{b^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{a^n+c^n}}{b^m+c^m}$$

$$+\frac{a^{2m}.\sqrt{a^n+b^n}+c^{2m}.\sqrt{b^n+c^n}}{a^m+c^m} \geq 3\sqrt{2}\ \forall\, a,b,c>0 \,\middle|\, abc=1$$

and $\forall$ m, n $\in \mathbb{N}$,'' = '' iff $a = b = c = 1$ (QED)