ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}}+\frac{\sqrt{b+c}.\left(b^2.\sqrt{ab}+c^2.\sqrt{ac}\right)}{b.\sqrt{a+b}+c.\sqrt{a+c}}$$

$$+\frac{\sqrt{a+c}.\left(a^2.\sqrt{ab}+c^2.\sqrt{bc}\right)}{a.\sqrt{a+b}+c.\sqrt{b+c}}\ge 3$$

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$$\frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}}\overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{CBS}}}{\ge}\frac{\sqrt{a+b}.\frac{1}{2}(b^2+a^2).\sqrt{c}.\left(\sqrt{b}+\sqrt{a}\right)}{\sqrt{b^2+a^2}.\sqrt{b+c+c+a}}$$

$$\overset{abc=1}{=}\frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\right)}{\sqrt{b+c+c+a}}=\frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)}{\sqrt{b+c+c+a}}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{4}{\sqrt{a}+\sqrt{b}}\right)}{\sqrt{b+c+c+a}}\overset{\text{CBS}}{\ge}\frac{\frac{1}{2}.\sqrt{a+b}.\sqrt{a^2+b^2}.\left(\frac{4}{\sqrt{2(a+b)}}\right)}{\sqrt{b+c+c+a}}$$

$$=\frac{\sqrt{2(a^2+b^2)}}{\sqrt{b+c+c+a}}\overset{\text{Reverse CBS}}{\ge}\frac{a+b}{\sqrt{b+c+c+a}}$$

$$\therefore\frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}}\ge\frac{a+b}{\sqrt{b+c+c+a}}\text{ and analogs}$$

$$\Rightarrow\frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}}+\frac{\sqrt{b+c}.\left(b^2.\sqrt{ab}+c^2.\sqrt{ac}\right)}{b.\sqrt{a+b}+c.\sqrt{a+c}}$$

$$+\frac{\sqrt{a+c}.\left(a^2.\sqrt{ab}+c^2.\sqrt{bc}\right)}{a.\sqrt{a+b}+c.\sqrt{b+c}}\ge\sum_{cyc}\frac{a+b}{\sqrt{b+c+c+a}}\overset{A-G}{\ge}3.\sqrt[3]{\frac{\prod_{cyc}(a+b)}{\prod_{cyc}\sqrt{b+c+c+a}}}$$

$$\overset{abc=1}{=}3.\sqrt[3]{\frac{\prod_{cyc}(a+b)}{\sqrt{abc}.\prod_{cyc}\sqrt{b+c+c+a}}}\overset{?}{\ge}3\Leftrightarrow\prod_{cyc}(a+b)^2\underbrace{\overset{?}{\ge}}_{(*)}abc\prod_{cyc}(b+c+c+a)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=(s-x)(s-y)(s-z)\Rightarrow abc=r^2s$ and via such substitutions, $(*)\Leftrightarrow$

$x^2y^2z^2\ge r^2s(x+y)(y+z)(z+x)\Leftrightarrow 16R^2r^2s^2\ge r^2s.2s(s^2+2Rr+r^2)$

$\Leftrightarrow s^2\le 8R^2-2Rr-r^2\Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r)\le 0\rightarrow$ true

$$\because s^2 - 4R^2 - 4Rr - 3r^2 \overset{\text{Gerretsen}}{\leq} 0 \text{ and } -2(2R+r)(R-2r) \overset{\text{Euler}}{\leq} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{\sqrt{a+b}.\left(b^2.\sqrt{bc}+a^2.\sqrt{ac}\right)}{b.\sqrt{b+c}+a.\sqrt{a+c}} + \frac{\sqrt{b+c}.\left(b^2.\sqrt{ab}+c^2.\sqrt{ac}\right)}{b.\sqrt{a+b}+c.\sqrt{a+c}}$$

$$+\frac{\sqrt{a+c}.\left(a^2.\sqrt{ab}+c^2.\sqrt{bc}\right)}{a.\sqrt{a+b}+c.\sqrt{b+c}} \geq 3 \ \forall \ a,b,c > 0 \,\middle|\, abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$