ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{a^2b^2.\sqrt{ab}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{b^2c^2.\sqrt{bc}+c^2a^2.\sqrt{ca}}+\frac{b^2c^2.\sqrt{bc}\left(\sqrt{a+b}+\sqrt{c+a}\right)}{a^2b^2.\sqrt{ab}+c^2a^2.\sqrt{ca}}$$
$$+\frac{c^2a^2.\sqrt{ca}\left(\sqrt{a+b}+\sqrt{b+c}\right)}{a^2b^2.\sqrt{ab}+b^2c^2.\sqrt{bc}}\geq 3\sqrt{2}$$

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$\forall\, A, B, C > 0$, $(A+B), (B+C), (C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$)$
$\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and

$$16F^2 = 2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2$$
$$= 2\sum_{cyc}\left(\sum_{cyc}AB + B^2\right) - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB$$
$$= 6\sum_{cyc}AB + 2\sum_{cyc}A^2 - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall\, x, y, z > 0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy + z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2 + xyz\sum_{cyc}x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (1)$$

$$\text{We have : } \frac{a^2b^2.\sqrt{ab}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{b^2c^2.\sqrt{bc}+c^2a^2.\sqrt{ca}}+\frac{b^2c^2.\sqrt{bc}\left(\sqrt{a+b}+\sqrt{c+a}\right)}{a^2b^2.\sqrt{ab}+c^2a^2.\sqrt{ca}}$$
$$+\frac{c^2a^2.\sqrt{ca}\left(\sqrt{a+b}+\sqrt{b+c}\right)}{a^2b^2.\sqrt{ab}+b^2c^2.\sqrt{bc}} = \frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(\begin{matrix} x = a^2b^2.\sqrt{ab}, y = b^2c^2.\sqrt{bc}, z = c^2a^2.\sqrt{ca}, \\ A = \sqrt{a+b}, B = \sqrt{b+c}, C = \sqrt{c+a}\end{matrix}\right)$$
$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB.\frac{\sqrt{3}}{2}}$$

$$= \sqrt{3\sum_{cyc}\sqrt{(a+b)(b+c)}} \overset{A-G}{\geq} \sqrt{9.\sqrt[3]{(a+b)(b+c)(c+a)}}$$

$$\overset{Cesaro}{\geq} 3.\sqrt{\sqrt[3]{8abc}} \overset{abc=1}{=} 3\sqrt{2}\ \forall a,b,c>0 \,\big|\, abc=1, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$