ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{\sqrt{(a+c)(b^3+c^3)}+\sqrt{(b+c)(a^3+c^3)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}+\frac{\sqrt{(a+c)(a^3+b^3)}+\sqrt{(a+b)(a^3+c^3)}}{\sqrt{b+c}\left(\sqrt{a+b}+\sqrt{a+c}\right)}$$

$$+\frac{\sqrt{(b+c)(a^3+b^3)}+\sqrt{(a+b)(b^3+c^3)}}{\sqrt{a+c}\left(\sqrt{a+b}+\sqrt{b+c}\right)}\geq 3$$

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Solution by Soumava Chakraborty-India

$$\frac{\sqrt{(a+c)(b^3+c^3)}+\sqrt{(b+c)(a^3+c^3)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}$$

$$=\frac{\sqrt{(c+a)(b+c)(b^2-bc+c^2)}+\sqrt{(b+c)(c+a)(c^2-ca+a^2)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}$$

$$\geq\frac{\sqrt{(b+c)(c+a)}\left(\frac{b+c}{2}+\frac{c+a}{2}\right)}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}=\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}^2+\sqrt{c+a}^2\right)}{2.\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}$$

$$\geq\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)^2}{4.\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{c+a}\right)}$$

$$\Rightarrow\frac{\sqrt{(a+c)(b^3+c^3)}+\sqrt{(b+c)(a^3+c^3)}}{\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{a+c}\right)}\geq\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{4.\sqrt{a+b}}$$

$$and\ analogs\Rightarrow \text{LHS}\geq\frac{1}{4}\sum_{cyc}\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{\sqrt{a+b}}\overset{A-G}{\geq}$$

$$\frac{3}{4}.\sqrt[3]{\frac{\left(\prod_{cyc}\sqrt{(b+c)(c+a)}\right)\left(\prod_{cyc}\left(\sqrt{b+c}+\sqrt{c+a}\right)\right)}{\prod_{cyc}\sqrt{a+b}}}$$

$$\overset{Cesaro}{\geq}\frac{3}{4}.\sqrt[3]{\sqrt{(a+b)(b+c)(c+a)}.8.\sqrt{(a+b)(b+c)(c+a)}}$$

$$=\frac{3}{2}.\sqrt[3]{(a+b)(b+c)(c+a)}\overset{Cesaro}{\geq}\frac{3}{2}.\sqrt[3]{8abc}\overset{abc=1}{=}3\ \forall a,b,c>0\,\middle|\,abc=1,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$