ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{b\sqrt{b}(b^5+c^5)+a\sqrt{a}(a^5+c^5)}{\sqrt{c}(a+b)}+\frac{b\sqrt{b}(a^5+b^5)+c\sqrt{c}(a^5+c^5)}{\sqrt{a}(b+c)}$$
$$+\frac{a\sqrt{a}(a^5+b^5)+c\sqrt{c}(b^5+c^5)}{\sqrt{b}(a+c)}\geq 6$$

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$$\text{LHS}=(b^5+c^5)\left(\frac{b\sqrt{b}}{\sqrt{c}(a+b)}+\frac{c\sqrt{c}}{\sqrt{b}(c+a)}\right)$$
$$+(c^5+a^5)\left(\frac{a\sqrt{a}}{\sqrt{c}(a+b)}+\frac{c\sqrt{c}}{\sqrt{a}(b+c)}\right)+(a^5+b^5)\left(\frac{b\sqrt{b}}{\sqrt{a}(b+c)}+\frac{a\sqrt{a}}{\sqrt{b}(a+c)}\right)\to(1)$$

$$\text{Now, }\frac{b\sqrt{b}}{\sqrt{c}(a+b)}+\frac{c\sqrt{c}}{\sqrt{b}(c+a)}=\frac{b^2}{\sqrt{bc}(a+b)}+\frac{c^2}{\sqrt{bc}(c+a)}$$
$$\overset{\text{Bergstrom}}{\geq}\frac{(b+c)^2}{\sqrt{bc}\big((c+a)+(a+b)\big)}\text{ and analogs}\to(2)\ \therefore(1)\text{ and }(2)\Rightarrow$$

$$\text{LHS}\geq\sum_{\text{cyc}}\left((b^5+c^5).\frac{(b+c)^2}{\sqrt{bc}\big((c+a)+(a+b)\big)}\right)\overset{A-G}{\geq}$$
$$3.\sqrt[3]{\left(\prod_{\text{cyc}}(b^5+c^5)\right)\frac{\prod_{\text{cyc}}(b+c)^2}{abc\prod_{\text{cyc}}\big((c+a)+(a+b)\big)}}$$
$$\overset{\text{Cesaro}}{\geq}3.\sqrt[3]{8(abc)^5.\frac{\prod_{\text{cyc}}(b+c)^2}{abc\prod_{\text{cyc}}\big((c+a)+(a+b)\big)}}$$
$$\overset{abc=1}{=}6.\sqrt[3]{\frac{\prod_{\text{cyc}}(b+c)^2}{abc\prod_{\text{cyc}}\big((c+a)+(a+b)\big)}}\overset{?}{\geq}6$$
$$\Leftrightarrow\prod_{\text{cyc}}(a+b)^2\underbrace{\overset{?}{\geq}}_{(*)}abc\prod_{\text{cyc}}\big((c+a)+(a+b)\big)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ *and* $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (*say*) yielding $2\sum_{\text{cyc}}a=\sum_{\text{cyc}}x=2s\Rightarrow\sum_{\text{cyc}}a=s\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=(s-x)(s-y)(s-z)\Rightarrow abc=r^2s$ *and via* such substitutions, $(*)\Leftrightarrow$

$$x^2y^2z^2\geq r^2s(x+y)(y+z)(z+x)\Leftrightarrow 16R^2r^2s^2\geq r^2s.2s(s^2+2Rr+r^2)$$
$$\Leftrightarrow s^2\leq 8R^2-2Rr-r^2\Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r)\leq 0\to\text{true}$$
$$\because s^2-4R^2-4Rr-3r^2\overset{\text{Gerretsen}}{\leq}0\text{ and }-2(2R+r)(R-2r)\overset{\text{Euler}}{\leq}0\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{b\sqrt{b}(b^5+c^5)+a\sqrt{a}(a^5+c^5)}{\sqrt{c}(a+b)}+\frac{b\sqrt{b}(a^5+b^5)+c\sqrt{c}(a^5+c^5)}{\sqrt{a}(b+c)}$$
$$+\frac{a\sqrt{a}(a^5+b^5)+c\sqrt{c}(b^5+c^5)}{\sqrt{b}(a+c)}\geq 6\ \forall\ a,b,c>0 \mid abc=1,$$
$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$