ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{\sqrt{a^3+b^3}\left(\sqrt[3]{b}.\sqrt[4]{b^5+c^5}+\sqrt[3]{a}.\sqrt[4]{c^5+a^5}\right)}{\sqrt[3]{bc}.\sqrt{b^3+c^3}+\sqrt[3]{ac}.\sqrt{c^3+a^3}}$$
$$+\frac{\sqrt{b^3+c^3}\left(\sqrt[3]{b}.\sqrt[4]{a^5+b^5}+\sqrt[3]{c}.\sqrt[4]{c^5+a^5}\right)}{\sqrt[3]{ab}.\sqrt{a^3+b^3}+\sqrt[3]{ac}.\sqrt{c^3+a^3}}$$
$$+\frac{\sqrt{c^3+a^3}\left(\sqrt[3]{a}.\sqrt[4]{a^5+b^5}+\sqrt[3]{c}.\sqrt[4]{b^5+c^5}\right)}{\sqrt[3]{ab}.\sqrt{a^3+b^3}+\sqrt[3]{bc}.\sqrt{b^3+c^3}}\geq 3.\sqrt[4]{2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A,B,C>0,(A+B),(B+C),(C+A) \text{ form sides of } a \text{ triangle}$$
$$(\because\ (A+B)+(B+C)>(C+A) \text{ and analogs})$$
$$\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A} \text{ form sides of } a \text{ triangle}$$
$$\text{with area } F\ (say) \text{ and } 16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$
$$\text{Now}, \forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of } (*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}$$
$$=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow \text{true}$$
$$\therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$
$$\text{We have : LHS}=\frac{\sqrt[3]{ab}.\sqrt{a^3+b^3}\left(\frac{\sqrt[4]{b^5+c^5}}{\sqrt[3]{a}}+\frac{\sqrt[4]{c^5+a^5}}{\sqrt[3]{b}}\right)}{\sqrt[3]{bc}.\sqrt{b^3+c^3}+\sqrt[3]{ca}.\sqrt{c^3+a^3}}$$

$$+\frac{\sqrt[3]{\mathbf{bc}}.\sqrt{\mathbf{b}^3+\mathbf{c}^3}\left(\frac{\sqrt[4]{a^5+\mathbf{b}^5}}{\sqrt[3]{\mathbf{c}}}+\frac{\sqrt[4]{\mathbf{c}^5+a^5}}{\sqrt[3]{\mathbf{b}}}\right)}{\sqrt[3]{a\mathbf{b}}.\sqrt{a^3+\mathbf{b}^3}+\sqrt[3]{\mathbf{c}a}.\sqrt{\mathbf{c}^3+a^3}}+\frac{\sqrt[3]{\mathbf{c}a}.\sqrt{\mathbf{c}^3+a^3}\left(\frac{\sqrt[4]{a^5+\mathbf{b}^5}}{\sqrt[3]{\mathbf{c}}}+\frac{\sqrt[4]{\mathbf{b}^5+\mathbf{c}^5}}{\sqrt[3]{a}}\right)}{\sqrt[3]{a\mathbf{b}}.\sqrt{a^3+\mathbf{b}^3}+\sqrt[3]{\mathbf{bc}}.\sqrt{\mathbf{b}^3+\mathbf{c}^3}}$$

$$=\frac{x}{\mathbf{y}+\mathbf{z}}(\mathbf{B}+\mathbf{C})+\frac{\mathbf{y}}{\mathbf{z}+x}(\mathbf{C}+\mathbf{A})+\frac{\mathbf{z}}{x+\mathbf{y}}(\mathbf{A}+\mathbf{B})$$

$$\left(\begin{array}{c} x=\sqrt[3]{a\mathbf{b}}.\sqrt{a^3+\mathbf{b}^3},\mathbf{y}=\sqrt[3]{\mathbf{bc}}.\sqrt{\mathbf{b}^3+\mathbf{c}^3},\mathbf{z}=\sqrt[3]{\mathbf{c}a}.\sqrt{\mathbf{c}^3+a^3}, \\ \mathbf{A}=\frac{\sqrt[4]{a^5+\mathbf{b}^5}}{\sqrt[3]{\mathbf{c}}},\mathbf{B}=\frac{\sqrt[4]{\mathbf{b}^5+\mathbf{c}^5}}{\sqrt[3]{a}},\mathbf{C}=\frac{\sqrt[4]{\mathbf{c}^5+a^5}}{\sqrt[3]{\mathbf{b}}} \end{array}\right)$$

$$=\frac{x}{\mathbf{y}+\mathbf{z}}.\sqrt{\mathbf{B}+\mathbf{C}}^2+\frac{\mathbf{y}}{\mathbf{z}+x}.\sqrt{\mathbf{C}+\mathbf{A}}^2+\frac{\mathbf{z}}{x+\mathbf{y}}.\sqrt{\mathbf{A}+\mathbf{B}}^2 \overset{\text{Oppenheim}}{\geq}$$

$$\mathbf{4F}.\sqrt{\sum_{\text{cyc}}\frac{x\mathbf{y}}{(\mathbf{y}+\mathbf{z})(\mathbf{z}+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{\text{cyc}}\mathbf{AB}.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{\text{cyc}}\frac{\sqrt[4]{(a^5+\mathbf{b}^5)(\mathbf{b}^5+\mathbf{c}^5)}}{\sqrt[3]{\mathbf{c}a}}}$$

$$\overset{\text{A-G}}{\geq}\sqrt{9.\sqrt[3]{\frac{\sqrt{(a^5+\mathbf{b}^5)(\mathbf{b}^5+\mathbf{c}^5)(\mathbf{c}^5+a^5)}}{\sqrt[3]{a^2\mathbf{b}^2\mathbf{c}^2}}}}\overset{\substack{\text{Cesaro} \\ \text{and} \\ \because abc=1}}{\geq}3.\sqrt{\sqrt[3]{\sqrt{8(a\mathbf{bc})^5}}}\overset{abc=1}{=}3.\sqrt[12]{2^3}$$

$$=3.\sqrt[4]{2}\ \ \forall a,\mathbf{b},\mathbf{c}>0\,\middle|\,a\mathbf{bc}=1,''=''\ \text{iff}\ a=\mathbf{b}=\mathbf{c}=1\ (\text{QED})$$