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If $a,\mathrm{b},\mathrm{c}>0$ and $a\mathrm{bc}=1$, then prove that :

$$\frac{\sqrt[5]{c}\left(b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}\right)}{b.\sqrt[5]{a}+a.\sqrt[5]{b}}+\frac{\sqrt[5]{a}\left(b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}\right)}{b.\sqrt[5]{c}+c.\sqrt[5]{b}}+\frac{\sqrt[5]{b}\left(a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}\right)}{a.\sqrt[5]{c}+c.\sqrt[5]{a}}\geq 3$$

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$$\forall\, A,B,C>0,\ (A+B),(B+C),(C+A) \text{ form sides of } a \text{ triangle}$$
$$(\because (A+B)+(B+C)>(C+A) \text{ and analogs})$$
$$\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A} \text{ form sides of } a \text{ triangle with area } F \text{ (say)}$$
$$\text{and } 16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now}, \forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\ \therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\sqrt[5]{c}\left(b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}\right)}{b.\sqrt[5]{a}+a.\sqrt[5]{b}}+\frac{\sqrt[5]{a}\left(b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}\right)}{b.\sqrt[5]{c}+c.\sqrt[5]{b}}+\frac{\sqrt[5]{b}\left(a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}\right)}{a.\sqrt[5]{c}+c.\sqrt[5]{a}}$$

$$=\frac{\frac{\sqrt[5]{a}}{a}\left(a.\frac{b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}}{bc}\right)}{\frac{b.\sqrt[5]{c}+c.\sqrt[5]{b}}{bc}}+\frac{\frac{\sqrt[5]{b}}{b}\left(b.\frac{a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}}{ca}\right)}{\frac{a.\sqrt[5]{c}+c.\sqrt[5]{a}}{ca}}+\frac{\frac{\sqrt[5]{c}}{c}\left(c.\frac{b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}}{ab}\right)}{\frac{b.\sqrt[5]{a}+a.\sqrt[5]{b}}{ab}}$$

$$\overset{abc=1}{=}\frac{\frac{\sqrt[5]{a}}{a}\left(\frac{b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}}{b^2c^2}\right)}{\frac{\sqrt[5]{b}}{b}+\frac{\sqrt[5]{c}}{c}}+\frac{\frac{\sqrt[5]{b}}{b}\left(\frac{a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}}{c^2a^2}\right)}{\frac{\sqrt[5]{c}}{c}+\frac{\sqrt[5]{a}}{a}}+\frac{\frac{\sqrt[5]{c}}{c}\left(\frac{b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}}{a^2b^2}\right)}{\frac{\sqrt[5]{a}}{a}+\frac{\sqrt[5]{b}}{b}}$$

$$=\frac{\frac{\sqrt[5]{a}}{a}\left(\frac{\sqrt[3]{b}}{b^2}+\frac{\sqrt[3]{c}}{c^2}\right)}{\frac{\sqrt[5]{b}}{b}+\frac{\sqrt[5]{c}}{c}}+\frac{\frac{\sqrt[5]{b}}{b}\left(\frac{\sqrt[3]{c}}{c^2}+\frac{\sqrt[3]{a}}{a^2}\right)}{\frac{\sqrt[5]{c}}{c}+\frac{\sqrt[5]{a}}{a}}+\frac{\frac{\sqrt[5]{c}}{c}\left(\frac{\sqrt[3]{a}}{a^2}+\frac{\sqrt[3]{b}}{b^2}\right)}{\frac{\sqrt[5]{a}}{a}+\frac{\sqrt[5]{b}}{b}}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

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$$\left(x=\frac{\sqrt[5]{a}}{a},y=\frac{\sqrt[5]{b}}{b},z=\frac{\sqrt[5]{c}}{c},A=\frac{\sqrt[3]{a}}{a^2},B=\frac{\sqrt[3]{b}}{b^2},C=\frac{\sqrt[3]{c}}{c^2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via }(1)\text{ and }(2)}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt[3]{a}}{a^2}.\frac{\sqrt[3]{b}}{b^2}\right)}\overset{A-G}{\geq}$$

$$\sqrt{9.\sqrt[3]{\frac{\sqrt[3]{a^2b^2c^2}}{a^4b^4c^4}}}\overset{abc=1}{=}\sqrt{9}=3\ \forall\, a,b,c>0\,\middle|\,abc=1,''='' \text{ iff } a=b=c=1\ (QED)$$