ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{c^4.\sqrt{bc}(c+a)^2 + b^4.\sqrt{bc}(a+b)^2}{c^2.\sqrt{ca} + b^2.\sqrt{ab}} \ge \frac{4}{3}.\sqrt{\frac{abc(a+b+c)}{3}}(a+b+c)^2$$

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$$\forall\, A, B, C > 0, (A+B), (B+C), (C+A) \text{ form sides of a triangle}$$
$$(\because (A+B)+(B+C) > (C+A) \text{ and analogs})$$

$\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \to (1)$$

$$\text{Now}, \forall\, x, y, z > 0, \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\ge} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\ge}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \ge \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\ge} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\ge} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\ge} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{c^4.\sqrt{bc}(c+a)^2 + b^4.\sqrt{bc}(a+b)^2}{c^2.\sqrt{ca} + b^2.\sqrt{ab}} + \frac{c^4.\sqrt{ca}(b+c)^2 + a^4.\sqrt{ca}(a+b)^2}{c^2.\sqrt{bc} + a^2.\sqrt{ab}}$$
$$+ \frac{b^4.\sqrt{ab}(b+c)^2 + a^4.\sqrt{ab}(c+a)^2}{b^2.\sqrt{bc} + a^2.\sqrt{ca}}$$

$$= \frac{\frac{c^2}{b^2}.\sqrt{bc}(c+a)^2 + \frac{b^2}{c^2}.\sqrt{bc}(a+b)^2}{\frac{\sqrt{ca}}{b^2} + \frac{\sqrt{ab}}{c^2}} + \frac{\frac{c^2}{a^2}.\sqrt{ca}(b+c)^2 + \frac{a^2}{c^2}.\sqrt{ca}(a+b)^2}{\frac{\sqrt{bc}}{a^2} + \frac{\sqrt{ab}}{c^2}}$$
$$+ \frac{\frac{b^2}{a^2}.\sqrt{ab}(b+c)^2 + \frac{a^2}{b^2}.\sqrt{ab}(c+a)^2}{\frac{\sqrt{bc}}{a^2} + \frac{\sqrt{ca}}{b^2}}$$

$$= \frac{\frac{\sqrt{bc}}{a^2}\left(\frac{c^2a^2(c+a)^2}{b^2} + \frac{a^2b^2(a+b)^2}{c^2}\right)}{\frac{\sqrt{ca}}{b^2} + \frac{\sqrt{ab}}{c^2}} + \frac{\frac{\sqrt{ca}}{b^2}\left(\frac{a^2b^2(a+b)^2}{c^2} + \frac{b^2c^2(b+c)^2}{a^2}\right)}{\frac{\sqrt{bc}}{a^2} + \frac{\sqrt{ab}}{c^2}}$$

$$+\frac{\frac{\sqrt{ab}}{c^2}\left(\frac{b^2c^2(b+c)^2}{a^2}+\frac{c^2a^2(c+a)^2}{b^2}\right)}{\frac{\sqrt{bc}}{a^2}+\frac{\sqrt{ca}}{b^2}}=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\frac{\sqrt{bc}}{a^2}, y=\frac{\sqrt{ca}}{b^2}, z=\frac{\sqrt{ab}}{c^2}, \\ A=\frac{b^2c^2(b+c)^2}{a^2}, B=\frac{c^2a^2(c+a)^2}{b^2}, C=\frac{a^2b^2(a+b)^2}{c^2}\end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{b^2c^2(b+c)^2}{a^2}.\frac{c^2a^2(c+a)^2}{b^2}\right)}=\sqrt{3\sum_{cyc}\left(c^4(c+a)^2(b+c)^2\right)}$$

$$\geq\sum_{cyc}\left(c^2(c+a)(b+c)\right)=\sum_{cyc}a^4+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)$$

$$\geq\frac{1}{3}\left(\sum_{cyc}a^2\right)^2+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)=\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)$$

$$\geq\frac{1}{9}\left(\sum_{cyc}a\right)^2\left(4\sum_{cyc}ab\right)\geq\frac{4}{9}\left(\sum_{cyc}a\right)^2\left(\sqrt{3abc\left(\sum_{cyc}a\right)}\right)$$

$$=\frac{4}{3}\left(\sum_{cyc}a\right)^2\left(\sqrt{\frac{3abc(\sum_{cyc}a)}{9}}\right)=\frac{4}{3}.\sqrt{\frac{abc(a+b+c)}{3}}(a+b+c)^2$$

$\forall\, a,b,c>0,'' ='' \text{ iff } a=b=c$ (QED)