ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+\frac{(\sqrt{c}+\sqrt{a})(c+a)}{\sqrt{c}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{a}+\sqrt{b})}$$
$$+\frac{(\sqrt{a}+\sqrt{b})(a+b)}{\sqrt{a}(\sqrt{c}+\sqrt{a})+\sqrt{b}(\sqrt{b}+\sqrt{c})}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A,B,C>0,(A+B),(B+C),(C+A) \text{ form sides of a triangle}$$
$$(\because (A+B)+(B+C)>(C+A) \text{ and analogs})$$
$$\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A} \text{ form sides of a triangle with area F (say) and } 16F^2=$$
$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$
$$\text{Now}, \forall x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of } (*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$
$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$
$$\text{We have : }\frac{(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+\frac{(\sqrt{c}+\sqrt{a})(c+a)}{\sqrt{c}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{a}+\sqrt{b})}$$
$$+\frac{(\sqrt{a}+\sqrt{b})(a+b)}{\sqrt{a}(\sqrt{c}+\sqrt{a})+\sqrt{b}(\sqrt{b}+\sqrt{c})}$$
$$\overset{abc=1}{=}\frac{\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)(b+c)}{\sqrt{bc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}+\frac{\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)(c+a)}{\sqrt{ca}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)}+\frac{\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)(a+b)}{\sqrt{ab}\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)}$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(x=\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}},y=\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}},z=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}},A=a,B=b,C=c\right)$$
$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\ge}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{via\ (1)\ and\ (2)}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}ab}\overset{A-G}{\geq}$$

$$\sqrt{3}.\sqrt{3.\sqrt[3]{a^2b^2c^2}}\overset{abc=1}{=}3\ \therefore\ \frac{(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+$$

$$\frac{(\sqrt{c}+\sqrt{a})(c+a)}{\sqrt{c}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{a}+\sqrt{b})}+\frac{(\sqrt{a}+\sqrt{b})(a+b)}{\sqrt{a}(\sqrt{c}+\sqrt{a})+\sqrt{b}(\sqrt{b}+\sqrt{c})}\geq 3$$

$$\forall\ a,b,c>0\ \middle|\ abc=1,''=''\ \text{iff}\ a=b=c=1\ (\text{QED})$$