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If $a, b, c, d > 0$, then prove that:

$$\frac{a}{2023b+2024c+2025d}+\frac{b}{2023c+2024d+2025a}+\frac{c}{2023d+2024a+2025b}+\frac{d}{2023a+2024b+2025c}\ge\frac{1}{1518}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{2023b+2024c+2025d}+\frac{b}{2023c+2024d+2025a}+\frac{c}{2023d+2024a+2025b}+\frac{d}{2023a+2024b+2025c}$$

$$=\frac{a^2}{2023ab+2024ac+2025ad}+\frac{b^2}{2023bc+2024bd+2025ab}+\frac{c^2}{2023cd+2024ca+2025bc}+\frac{d^2}{2023ad+2024bd+2025cd}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{(a+b+c+d)^2}{4048(ab+ac+ad+bc+bd+cd)}\overset{?}{\ge}\frac{1}{1518}$$

$$\Leftrightarrow\frac{a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)}{(ab+ac+ad+bc+bd+cd)}\overset{?}{\ge}\frac{8}{3}$$

$$\Leftrightarrow 3\left(a^2+b^2+c^2+d^2\right)\underset{(*)}{\overset{?}{\ge}}2(ab+ac+ad+bc+bd+cd)$$

Now, $a^2+b^2+c^2\ge ab+ac+bc\to(1)$, $a^2+b^2+d^2\ge ab+ad+bd\to(2)$,

$a^2+c^2+d^2\ge ac+ad+cd\to(3)$, $b^2+c^2+d^2\ge bc+bd+cd\to(4)$

$(1)+(2)+(3)+(4)\Rightarrow 3\left(a^2+b^2+c^2+d^2\right)\ge 2(ab+ac+ad+bc+bd+cd)$

$$\Rightarrow(*)\text{ is true }\therefore\frac{a}{2023b+2024c+2025d}+\frac{b}{2023c+2024d+2025a}+\frac{c}{2023d+2024a+2025b}+\frac{d}{2023a+2024b+2025c}\ge\frac{1}{1518}$$

$\forall\, a,b,c,d>0$, " = " iff $a=b=c=d$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Moroccco

By AM − GM inequality, we have

$$\frac{a}{2023b+2024c+2025d}+\frac{a(2023b+2024c+2025d)}{1518^2(a+b+c+d)^2}\ge\frac{a}{759(a+b+c+d)},$$

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then

$$\frac{a}{2023b+2024c+2025d} \geq \frac{a}{759(a+b+c+d)} - \frac{a(2023b+2024c+2025d)}{1518^2(a+b+c+d)^2}$$

(and analogs)

Therefore

$$\sum_{cyc} \frac{a}{2023b+2024c+2025d} \geq \frac{1}{759} - \frac{4048(ab+bc+cd+da+ac+bd)}{1518^2(a+b+c+d)^2}$$

$$\overset{Maclaurin}{\geq} \frac{1}{759} - \frac{4048}{1518^2} \cdot \frac{3}{8} = \frac{1}{1518},$$

as desired. Equality holds iff $a=b=c=d$.