ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, *then prove that* :

$$\frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{a(b+c)}+\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{b(a+c)}$$
$$+\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{c(a+b)}\geq 6$$

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$\forall\, A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle $(\because (A+B)+(B+C) > (C+A)$ *and analogs*$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with *area* F (*say*) *and* $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB}\rightarrow (1)$$

Now, $\forall\, x, y, z > 0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow \text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

We have : $\frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{a(b+c)}+\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{b(a+c)}$
$+\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{c(a+b)}$

$$=\frac{bc}{ca+ab}\cdot\frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{bc}$$
$$+\frac{ca}{ab+bc}\cdot\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{ca}$$
$$+\frac{ab}{bc+ca}\cdot\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{ab}$$

$$=\frac{bc\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)}{ca+ab}+\frac{ca\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)}{ab+bc}+\frac{ab\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}{bc+ca}$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x = bc, y = ca, z = ab, A = \frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}, B = \frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}, C = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB.\frac{\sqrt{3}}{2}} = \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}$$

$$\overset{A-G}{\geq} \sqrt{3}.\sqrt{3.\sqrt[3]{\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}.\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}.\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)}}$$

$$= \sqrt{3}.\sqrt{3.\left(\frac{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}{\sqrt{a}.\sqrt{b}.\sqrt{c}}\right)^{\frac{2}{3}}} \overset{\text{Cesaro}}{\geq} \sqrt{3}.\sqrt{3.8^{\frac{2}{3}}} = 6$$

$$\therefore \frac{b\sqrt{c}(\sqrt{a}+\sqrt{b}) + c\sqrt{b}(\sqrt{c}+\sqrt{a})}{a(b+c)} + \frac{c\sqrt{a}(\sqrt{b}+\sqrt{c}) + a\sqrt{c}(\sqrt{a}+\sqrt{b})}{b(a+c)}$$

$$+ \frac{b\sqrt{a}(\sqrt{b}+\sqrt{c}) + a\sqrt{b}(\sqrt{c}+\sqrt{a})}{c(a+b)} \geq 6\ \forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c\ (\text{QED})$$