ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that:

$$\frac{\sqrt{\frac{c}{a}+\frac{b}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a}{b}+\frac{c}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}}+\frac{\sqrt{\frac{b}{c}+\frac{a}{c}}}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}\ge 3$$

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$\forall\, A,B,C>0,\ (A+B),(B+C),(C+A)$ form sides of a triangle $(\because\ (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have: }\frac{\sqrt{\frac{c}{a}+\frac{b}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a}{b}+\frac{c}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}}+\frac{\sqrt{\frac{b}{c}+\frac{a}{c}}}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}$$

$$=\frac{c.\sqrt{\frac{b}{a}+\frac{b}{a}}.\sqrt{bc}}{b+c}+\frac{a.\sqrt{\frac{c}{b}+\frac{c}{b}}.\sqrt{ca}}{c+a}+\frac{b.\sqrt{\frac{a}{c}+\frac{a}{c}}.\sqrt{ab}}{a+b}$$

$$=\frac{\sqrt{\frac{abc}{a^2}}\left(\sqrt{c}+\frac{b}{\sqrt{a}}\right)}{b+c}+\frac{\sqrt{\frac{abc}{b^2}}\left(\sqrt{a}+\frac{c}{\sqrt{b}}\right)}{c+a}+\frac{\sqrt{\frac{abc}{c^2}}\left(\sqrt{b}+\frac{a}{\sqrt{c}}\right)}{a+b}$$

$$=\frac{c.\sqrt{ab}+b.\sqrt{bc}}{ab+ca}+\frac{a.\sqrt{bc}+c.\sqrt{ca}}{bc+ab}+\frac{b.\sqrt{ca}+a.\sqrt{ab}}{ca+bc}$$

$$=bc.\frac{\frac{c.\sqrt{ab}+b.\sqrt{bc}}{bc}}{ab+ca}+ca.\frac{\frac{a.\sqrt{bc}+c.\sqrt{ca}}{ca}}{bc+ab}+ab.\frac{\frac{b.\sqrt{ca}+a.\sqrt{ab}}{ab}}{ca+bc}$$

$$=\frac{bc\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}\right)}{ca+ab}+\frac{ca\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}\right)}{ab+bc}+\frac{ab\left(\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{b}}\right)}{bc+ca}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=bc, y=ca, z=ab, A=\sqrt{\frac{c}{a}}, B=\sqrt{\frac{a}{b}}, C=\sqrt{\frac{b}{c}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{c}{a}}.\sqrt{\frac{a}{b}}\right)}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{c}{b}}\right)} \overset{A-G}{\geq} \sqrt{3}.\sqrt{3.\sqrt[3]{\sqrt{\frac{c}{b}}.\sqrt{\frac{b}{a}}.\sqrt{\frac{a}{c}}}}=3$$

$$\therefore \frac{\sqrt{\frac{c}{a}+\frac{b}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a}{b}+\frac{c}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}}+\frac{\sqrt{\frac{b}{c}+\frac{a}{c}}}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}\geq 3\ \forall\, a,b,c>0, ''='' \text{ iff } a=b=c\ (QED)$$