ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that:

$$\frac{\sqrt{c(a+c)}+\sqrt{b(a+b)}}{a\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+\frac{\sqrt{c(b+c)}+\sqrt{a(a+b)}}{b\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}+\frac{\sqrt{b(b+c)}+\sqrt{a(a+c)}}{c\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}\geq 3\sqrt{2}$$

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$\forall A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle $(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now, } \forall x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

We have:

$$\frac{\sqrt{c(a+c)}+\sqrt{b(a+b)}}{a\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+\frac{\sqrt{c(b+c)}+\sqrt{a(a+b)}}{b\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}+\frac{\sqrt{b(b+c)}+\sqrt{a(a+c)}}{c\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}$$

$$=\frac{\sqrt{bc}\left(\sqrt{c(a+c)}+\sqrt{b(a+b)}\right)}{a(b+c)}+\frac{\sqrt{ca}\left(\sqrt{c(b+c)}+\sqrt{a(a+b)}\right)}{b(a+c)}+\frac{\sqrt{ab}\left(\sqrt{b(b+c)}+\sqrt{a(a+c)}\right)}{c(a+b)}$$

$$=\frac{bc}{ca+ab}\cdot\left(\sqrt{\frac{c(a+c)}{bc}}+\sqrt{\frac{b(a+b)}{bc}}\right)+\frac{ca}{ab+bc}\cdot\left(\sqrt{\frac{c(b+c)}{ca}}+\sqrt{\frac{a(a+b)}{ca}}\right)$$

$$+\frac{ab}{bc+ca}\cdot\left(\sqrt{\frac{b(b+c)}{ab}}+\sqrt{\frac{a(a+c)}{ab}}\right)$$

$$=\frac{bc}{ca+ab}\cdot\left(\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}}\right)+\frac{ca}{ab+bc}\cdot\left(\sqrt{\frac{b+c}{a}}+\sqrt{\frac{a+b}{c}}\right)$$

$$+\frac{ab}{bc+ca}\cdot\left(\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=bc,y=ca,z=ab,A=\sqrt{\frac{b+c}{a}},B=\sqrt{\frac{c+a}{b}},C=\sqrt{\frac{a+b}{c}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\sqrt{\frac{b+c}{a}}.\sqrt{\frac{c+a}{b}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3.3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}}\overset{\text{Cesaro}}{\geq}\sqrt{3.3.\sqrt[3]{8}}=3\sqrt{2}$$

$$\therefore\frac{\sqrt{c(a+c)}+\sqrt{b(a+b)}}{a\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+\frac{\sqrt{c(b+c)}+\sqrt{a(a+b)}}{b\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}+\frac{\sqrt{b(b+c)}+\sqrt{a(a+c)}}{c\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}$$

$$\geq 3\sqrt{2}\ \forall\ a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$