ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\sum_{cyc} \frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}} \geq 3\sqrt{a+b+c}.$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle

$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \rightarrow (1)$$

$$\text{Now}, \forall x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have : } \frac{\sqrt{b(b^2+bc+c^2)}+\sqrt{a(a^2+ac+c^2)}}{\sqrt{ac}+\sqrt{bc}}$$

$$+\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}} + \frac{\sqrt{a(a^2+ab+b^2)}+\sqrt{c(b^2+bc+c^2)}}{\sqrt{ab}+\sqrt{bc}}$$

$$= \frac{\sqrt{ab}}{\sqrt{bc}+\sqrt{ca}}\cdot\left(\sqrt{\frac{b^2+bc+c^2}{a}}+\sqrt{\frac{c^2+ca+a^2}{b}}\right)$$

$$+\frac{\sqrt{bc}}{\sqrt{ca}+\sqrt{ab}}\cdot\left(\sqrt{\frac{c^2+ca+a^2}{b}}+\sqrt{\frac{a^2+ab+b^2}{c}}\right)$$

$$+\frac{\sqrt{ca}}{\sqrt{ab}+\sqrt{bc}}.\left(\sqrt{\frac{a^2+ab+b^2}{c}}+\sqrt{\frac{b^2+bc+c^2}{a}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\sqrt{ab}, y=\sqrt{bc}, z=\sqrt{ca}, \\ A=\sqrt{\frac{a^2+ab+b^2}{c}}, B=\sqrt{\frac{b^2+bc+c^2}{a}}, C=\sqrt{\frac{c^2+ca+a^2}{b}} \end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{a^2+ab+b^2}{c}}.\sqrt{\frac{b^2+bc+c^2}{a}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{3ab}{c}}.\sqrt{\frac{3bc}{a}}\right)}=\sqrt{3}.\sqrt{3\sum_{cyc}b}=3.\sqrt{a+b+c}$$

$$\therefore \frac{\sqrt{b(b^2+bc+c^2)}+\sqrt{a(a^2+ac+c^2)}}{\sqrt{ac}+\sqrt{bc}}+\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}}$$

$$+\frac{\sqrt{a(a^2+ab+b^2)}+\sqrt{c(b^2+bc+c^2)}}{\sqrt{ab}+\sqrt{bc}}\geq 3.\sqrt{a+b+c}$$

$$\forall\, a,b,c>0,'' ='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\sqrt{a}, y:=\sqrt{b}, z:=\sqrt{c}$. **We have**

$$\sum_{cyc}\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}} \overset{AM-GM}{\geq} \sum_{cyc}\frac{\sqrt{b.3ab}+\sqrt{c.3ac}}{\sqrt{ab}+\sqrt{ac}}$$

$$=\sqrt{3}.\sum_{cyc}\frac{y^2+z^2}{y+z} \overset{CBS}{\geq} \sqrt{3}.\frac{\left(\sum_{cyc}\sqrt{y^2+z^2}\right)^2}{2(x+y+z)}$$

$$=\sqrt{3}.\frac{2\sum_{cyc}x^2+2\sum_{cyc}\sqrt{(x^2+y^2)(x^2+z^2)}}{2(x+y+z)}$$

$$\overset{CBS}{\geq}\sqrt{3}.\frac{2\sum_{cyc}x^2+2\sum_{cyc}(x^2+yz)}{2(x+y+z)}=\sqrt{3}.\frac{3\sum_{cyc}x^2+\left(\sum_{cyc}x\right)^2}{2(x+y+z)}$$

$$\overset{AM-GM}{\geq}3\sqrt{x^2+y^2+z^2}=3\sqrt{a+b+c}.$$

Equality holds iff $a=b=c$.