ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a.\sqrt{b^3+c^3}+b.\sqrt{a^3+c^3}}{ab(a+b)}+\frac{c.\sqrt{a^3+b^3}+b.\sqrt{a^3+c^3}}{bc(b+c)}+\frac{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}{ac(a+c)}\ge 3\sqrt{2}$$

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$\forall$ A, B, C >0, $(A+B),(B+C),(C+A)$ form sides of a triangle ($\because (A+B)+(B+C)>(C+A)$ and analogs) $\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now}, \forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{a.\sqrt{b^3+c^3}+b.\sqrt{a^3+c^3}}{ab(a+b)}+\frac{c.\sqrt{a^3+b^3}+b.\sqrt{a^3+c^3}}{bc(b+c)}+\frac{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}{ac(a+c)}$$

$$\overset{abc=1}{=}\frac{ac.\sqrt{b^3+c^3}+bc.\sqrt{a^3+c^3}}{a+b}+\frac{ca.\sqrt{a^3+b^3}+ab.\sqrt{a^3+c^3}}{b+c}+\frac{bc.\sqrt{a^3+b^3}+ab.\sqrt{b^3+c^3}}{c+a}$$

$$=\frac{c}{a+b}.\left(a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}\right)+\frac{a}{b+c}.\left(c.\sqrt{a^3+b^3}+b.\sqrt{c^3+a^3}\right)+\frac{b}{c+a}.\left(c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}\right)$$

$$=\frac{a}{b+c}.\left(b.\sqrt{c^3+a^3}+c.\sqrt{a^3+b^3}\right)+\frac{b}{c+a}.\left(c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}\right)+\frac{c}{a+b}.\left(a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x = a, y = b, z = z, A = a.\sqrt{b^3 + c^3}, B = b.\sqrt{c^3 + a^3}, C = c.\sqrt{a^3 + b^3}\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB.\frac{\sqrt{3}}{2}}$$

$$= \sqrt{3}.\sqrt{\sum_{cyc}\left(a.\sqrt{b^3+c^3}.b.\sqrt{c^3+a^3}\right)} \overset{A-G}{\geq} \sqrt{3}.\sqrt{3.\sqrt[3]{a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3)}}$$

$$\overset{Cesaro}{\geq} 3.\sqrt[6]{a^2b^2c^2.8a^3b^3c^3} \overset{abc=1}{=} 3\sqrt{2} \therefore \frac{a.\sqrt{b^3+c^3} + b.\sqrt{a^3+c^3}}{ab(a+b)}$$

$$+ \frac{c.\sqrt{a^3+b^3} + b.\sqrt{a^3+c^3}}{bc(b+c)} + \frac{c.\sqrt{a^3+b^3} + a.\sqrt{b^3+c^3}}{ac(a+c)} \geq 3\sqrt{2}$$

$$\forall\, a, b, c > 0 \,\middle|\, abc = 1, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$