ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{(a+b).\sqrt{a^3+b^3}}{a^2b.\sqrt{b^3+c^3}+b^2a.\sqrt{c^3+a^3}}+\frac{(b+c).\sqrt{b^3+c^3}}{c^2b.\sqrt{a^3+b^3}+b^2c.\sqrt{c^3+a^3}}+\frac{(c+a).\sqrt{c^3+a^3}}{c^2a.\sqrt{a^3+b^3}+a^2c.\sqrt{b^3+c^3}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, $(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB}\rightarrow (1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{(a+b).\sqrt{a^3+b^3}}{a^2b.\sqrt{b^3+c^3}+b^2a.\sqrt{c^3+a^3}}+\frac{(b+c).\sqrt{b^3+c^3}}{c^2b.\sqrt{a^3+b^3}+b^2c.\sqrt{c^3+a^3}}$$

$$+\frac{(c+a).\sqrt{c^3+a^3}}{c^2a.\sqrt{a^3+b^3}+a^2c.\sqrt{b^3+c^3}}$$

$$=\frac{\left(\frac{a+b}{ab}\right).\sqrt{a^3+b^3}}{a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}}+\frac{\left(\frac{b+c}{bc}\right).\sqrt{b^3+c^3}}{c.\sqrt{a^3+b^3}+b.\sqrt{c^3+a^3}}$$

$$+\frac{\left(\frac{c+a}{ca}\right).\sqrt{c^3+a^3}}{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}\overset{abc=1}{=}\frac{c(a+b).\sqrt{a^3+b^3}}{a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}}$$

$$+\frac{a(b+c).\sqrt{b^3+c^3}}{c.\sqrt{a^3+b^3}+b.\sqrt{c^3+a^3}}+\frac{b(c+a).\sqrt{c^3+a^3}}{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}$$

$$=\frac{a.\sqrt{b^3+c^3}}{b.\sqrt{c^3+a^3}+c.\sqrt{a^3+b^3}}.(b+c)+\frac{b.\sqrt{c^3+a^3}}{c.\sqrt{a^3+b^3}+a.\sqrt{b^3+c^3}}.(c+a)+$$

$$\frac{c.\sqrt{a^3+b^3}}{a.\sqrt{b^3+c^3}+b.\sqrt{c^3+a^3}}.(a+b)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\left(x = a.\sqrt{b^3+c^3}, y = b.\sqrt{c^3+a^3}, z = c.\sqrt{a^3+b^3}, A = a, B = b, C = c\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}} = \sqrt{3\sum_{cyc}ab}$$

$$\overset{A-G}{\geq} \sqrt{9.\sqrt[3]{a^2b^2c^2}} \overset{abc=1}{=} 3$$

$$\therefore \frac{(a+b).\sqrt{a^3+b^3}}{a^2b.\sqrt{b^3+c^3}+b^2a.\sqrt{c^3+a^3}} + \frac{(b+c).\sqrt{b^3+c^3}}{c^2b.\sqrt{a^3+b^3}+b^2c.\sqrt{c^3+a^3}}$$

$$+ \frac{(c+a).\sqrt{c^3+a^3}}{c^2a.\sqrt{a^3+b^3}+a^2c.\sqrt{b^3+c^3}} \geq 3 \ \forall\, a,b,c > 0 \,\big|\, abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$