ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that:

$$\sum_{cyc}\sqrt{\frac{a+b}{c}}\cdot\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{b(b+c)}+\sqrt{a(c+a)}} \ge 6$$

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$\forall A, B, C > 0$, $(A+B), (B+C), (C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB + B^2\right) - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB + 2\sum_{cyc}A^2 - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now, } \forall x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\ge} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\ge}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \ge \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy + z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2 + xyz\sum_{cyc}x}$$

$$\overset{?}{\ge} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\ge} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\ge} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have: } \sqrt{\frac{a+b}{c}}\cdot\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{b(b+c)}+\sqrt{a(c+a)}} +$$

$$\sqrt{\frac{b+c}{a}}\cdot\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}{\sqrt{b(a+b)}+\sqrt{c(c+a)}} + \sqrt{\frac{a+c}{b}}\cdot\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}{\sqrt{a(a+b)}+\sqrt{c(b+c)}}$$

$$= \sqrt{\frac{a+b}{c}}\cdot\frac{\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{ab}}}{\sqrt{\frac{b(b+c)}{ab}}+\sqrt{\frac{a(c+a)}{ab}}} + \sqrt{\frac{b+c}{a}}\cdot\frac{\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}{\sqrt{bc}}}{\sqrt{\frac{b(a+b)}{bc}}+\sqrt{\frac{c(c+a)}{bc}}}$$

$$+ \sqrt{\frac{a+c}{b}}\cdot\frac{\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}{\sqrt{ca}}}{\sqrt{\frac{a(a+b)}{ca}}+\sqrt{\frac{c(b+c)}{ca}}}$$

$$= \frac{\sqrt{\frac{b+c}{a}}}{\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}}}\cdot\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right) + \frac{\sqrt{\frac{c+a}{b}}}{\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}}\cdot\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)$$

$$+\frac{\sqrt{\frac{a+b}{c}}}{\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}}.\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\sqrt{\frac{b+c}{a}}, y=\sqrt{\frac{c+a}{b}}, z=\sqrt{\frac{a+b}{c}}, \\ A=\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}, B=\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}, C=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}} \end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{via\ (1)\ and\ (2)}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}}=3.\sqrt[6]{\frac{(\sqrt{b}+\sqrt{c})^2(\sqrt{c}+\sqrt{a})^2(\sqrt{a}+\sqrt{b})^2}{abc}}$$

$$\overset{A-G}{\geq}3.\sqrt[6]{\frac{(4\sqrt{bc})(4\sqrt{ca})(4\sqrt{ab})}{abc}}=3.2=6$$

$$\therefore\sqrt{\frac{a+b}{c}}.\frac{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}{\sqrt{b(b+c)}+\sqrt{a(c+a)}}+\sqrt{\frac{b+c}{a}}.\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}{\sqrt{b(a+b)}+\sqrt{c(c+a)}}$$

$$+\sqrt{\frac{a+c}{b}}.\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}{\sqrt{a(a+b)}+\sqrt{c(b+c)}}\geq 6\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$