ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that:

$$\sum_{cyc} \frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}} \cdot \frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})} \geq 3\sqrt{2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}} \cdot \frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})} +$$

$$\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}} \cdot \frac{\sqrt{b(a+b)}+\sqrt{c(c+a)}}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})} + \frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a(a+b)}+\sqrt{c(b+c)}}{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})} \geq$$

$$\overset{\text{CBS}}{\geq} \frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}} \cdot \frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b}.\sqrt{2(b+c)}+\sqrt{a}.\sqrt{2(c+a)}} + \frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}} \cdot \frac{\sqrt{b(a+b)}+\sqrt{c(c+a)}}{\sqrt{b}.\sqrt{2(a+b)}+\sqrt{c}.\sqrt{2(c+a)}}$$

$$+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a(a+b)}+\sqrt{c(b+c)}}{\sqrt{a}.\sqrt{2(a+b)}+\sqrt{c}.\sqrt{2(b+c)}} =$$

$$= \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2c}} \cdot \frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b(b+c)}+\sqrt{a(c+a)}} + \frac{\sqrt{b}+\sqrt{c}}{\sqrt{2a}} \cdot \frac{\sqrt{b(a+b)}+\sqrt{c(c+a)}}{\sqrt{b(a+b)}+\sqrt{c(c+a)}}$$

$$+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{2b}} \cdot \frac{\sqrt{a(a+b)}+\sqrt{c(b+c)}}{\sqrt{a(a+b)}+\sqrt{c(b+c)}} \overset{A-G}{\geq}$$

$$\geq 3.\sqrt[3]{\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2c}} \cdot \frac{\sqrt{b}+\sqrt{c}}{\sqrt{2a}} \cdot \frac{\sqrt{c}+\sqrt{a}}{\sqrt{2b}}} \overset{\text{Cesaro}}{\geq} 3.\sqrt[3]{\frac{8(\sqrt{abc})}{2\sqrt{2}.(\sqrt{abc})}} = 3\sqrt{2}$$

$\forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c = (\text{QED})$