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If $a, b, c > 0$, *then prove that* :

$$\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\cdot\frac{\sqrt{a(b^2+c^2)}+\sqrt{b(c^2+a^2)}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)\geq 3\sqrt{2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\cdot\frac{\sqrt{a(b^2+c^2)}+\sqrt{b(c^2+a^2)}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)=$$

$$=\sum_{cyc}\left(\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)\cdot\left(\sqrt{\frac{b^2+c^2}{b}}+\sqrt{\frac{c^2+a^2}{a}}\right)\right)=$$

$$=\sum_{cyc}\left(\left(\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)\cdot\left(\sqrt{\frac{b}{c}+\frac{c}{b}}+\sqrt{\frac{c}{a}+\frac{a}{c}}\right)\right)\geq$$

$$\overset{A-G}{\geq} 2\sqrt{2}.\sum_{cyc}\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}=2\sqrt{2}.\sum_{cyc}\frac{x}{y+z}$$

$$\left(x=\sqrt{c}(\sqrt{a}+\sqrt{b}), y=\sqrt{a}(\sqrt{b}+\sqrt{c}), z=\sqrt{b}(\sqrt{c}+\sqrt{a})\right)\overset{Nesbitt}{\geq} 2\sqrt{2}.\frac{3}{2}=3\sqrt{2}$$

$$\therefore\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\cdot\frac{\sqrt{a(b^2+c^2)}+\sqrt{b(c^2+a^2)}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)\geq 3\sqrt{2}\ \forall\ a,b,c>0,$$

$$''='' \text{ iff } a=b=c\ (\text{QED})$$