ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c, d > 0$ and $n \in \mathbb{N}$, then prove that:

$$\frac{a}{nb+(n+1)c+(n+2)d}+\frac{b}{nc+(n+1)d+(n+2)a}+\frac{c}{nd+(n+1)a+(n+2)b}+\frac{d}{na+(n+1)b+(n+2)c}\ge\frac{4}{3(n+1)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{nb+(n+1)c+(n+2)d}+\frac{b}{nc+(n+1)d+(n+2)a}+\frac{c}{nd+(n+1)a+(n+2)b}+\frac{d}{na+(n+1)b+(n+2)c}$$

$$=\frac{a^2}{nab+(n+1)ac+(n+2)ad}+\frac{b^2}{nbc+(n+1)bd+(n+2)ab}+\frac{c^2}{ncd+(n+1)ca+(n+2)bc}+\frac{d^2}{nad+(n+1)bd+(n+2)cd}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{(a+b+c+d)^2}{2(n+1)(ab+ac+ad+bc+bd+cd)}\overset{?}{\ge}\frac{4}{3(n+1)}$$

$$\Leftrightarrow\frac{a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)}{ab+ac+ad+bc+bd+cd}\overset{?}{\ge}\frac{8}{3}$$

$$\Leftrightarrow 3\left(a^2+b^2+c^2+d^2\right)\underset{(*)}{\overset{?}{\ge}}2(ab+ac+ad+bc+bd+cd)$$

Now, $a^2+b^2+c^2\ge ab+ac+bc\to(1)$, $a^2+b^2+d^2\ge ab+ad+bd\to(2)$,
$a^2+c^2+d^2\ge ac+ad+cd\to(3)$, $b^2+c^2+d^2\ge bc+bd+cd\to(4)$
$(1)+(2)+(3)+(4)\Rightarrow 3\left(a^2+b^2+c^2+d^2\right)\ge 2(ab+ac+ad+bc+bd+cd)$

$$\Rightarrow (*)\text{ is true }\therefore\frac{a}{nb+(n+1)c+(n+2)d}+\frac{b}{nc+(n+1)d+(n+2)a}+\frac{c}{nd+(n+1)a+(n+2)b}+\frac{d}{na+(n+1)b+(n+2)c}\ge\frac{4}{3(n+1)}$$

$\forall\, a,b,c,d>0,$ "$=$" iff $a=b=c=d$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{a}{nb+(n+1)c+(n+2)d}+\frac{16a[nb+(n+1)c+(n+2)d]}{9(n+1)^2(a+b+c+d)^2}\ge\frac{8a}{3(n+1)(a+b+c+d)},$$

then

$$\frac{a}{nb+(n+1)c+(n+2)d}\ge\frac{8a}{3(n+1)(a+b+c+d)}-\frac{16[nab+(n+1)ac+(n+2)ad]}{9(n+1)^2(a+b+c+d)^2}$$

Adding this inequality with the similar ones, we obtain

$$\sum_{cyc} \frac{a}{nb+(n+1)c+(n+2)d} \geq \frac{8}{3(n+1)} - \frac{32}{9(n+1)} \cdot \frac{ab+bc+cd+da+ac+bd}{(a+b+c+d)^2}$$

$$\overset{Maclaurin}{\geq} \frac{8}{3(n+1)} - \frac{32}{9(n+1)} \cdot \frac{3}{8} = \frac{4}{3(n+1)},$$

as desired. Equality holds iff $a=b=c=d$.