If $a, \mathrm{b}, \mathrm{c} > 0$, then prove that :

$$\sum_{cyc}\sqrt{\frac{b+c}{a+b}}\cdot\frac{\sqrt{b+c}+\sqrt{a+c}}{\sqrt{(a+b)(b+c)}+(a+c)} \geq \frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle

$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form

sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB + B^2\right) - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB + 2\sum_{cyc}A^2 - 2\sum_{cyc}A^2 - 2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now}, \forall X, Y, Z > 0, \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{X^2Y^2}{XY(Y+Z)(Z+X)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}XY\right)^2}{\sum_{cyc}\left(XY\left(\sum_{cyc}XY + Z^2\right)\right)} = \frac{\left(\sum_{cyc}XY\right)^2}{\left(\sum_{cyc}XY\right)^2 + XYZ\sum_{cyc}X}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc}XY\right)^2 \overset{?}{\geq} 3XYZ\sum_{cyc}X \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have : } \sqrt{\frac{b+c}{a+b}}\cdot\frac{\sqrt{b+c}+\sqrt{a+c}}{\sqrt{(a+b)(b+c)}+(a+c)} +$$

$$\sqrt{\frac{a+c}{b+c}}\cdot\frac{\sqrt{a+c}+\sqrt{a+b}}{\sqrt{(b+c)(a+c)}+(a+b)} + \sqrt{\frac{a+b}{a+c}}\cdot\frac{\sqrt{a+b}+\sqrt{b+c}}{\sqrt{(a+c)(a+b)}+(b+c)}$$

$$= \frac{x}{z}\cdot\frac{x+y}{zx+y^2} + \frac{y}{x}\cdot\frac{y+z}{xy+z^2} + \frac{z}{y}\cdot\frac{z+x}{yz+x^2} \quad \left(x = \sqrt{b+c}, y = \sqrt{c+a}, z = \sqrt{a+b}\right)$$

$$= \frac{x^2y(x+y)}{xyz(zx+y^2)} + \frac{y^2z(y+z)}{yzx(xy+z^2)} + \frac{z^2x(z+x)}{zxy(yz+x^2)}$$

$$= \frac{x^2y}{z^2x+y^2z}\cdot\left(\frac{x+y}{xy}\right) + \frac{y^2z}{x^2y+z^2x}\cdot\left(\frac{y+z}{yz}\right) + \frac{z^2x}{y^2z+x^2y}\cdot\left(\frac{z+x}{zx}\right)$$

$$= \frac{y^2z}{z^2x+x^2y}\cdot\left(\frac{1}{y}+\frac{1}{z}\right) + \frac{z^2x}{x^2y+y^2z}\cdot\left(\frac{1}{z}+\frac{1}{x}\right) + \frac{x^2y}{y^2z+z^2x}\cdot\left(\frac{1}{x}+\frac{1}{y}\right)$$

$$=\frac{X}{Y+Z}(B+C)+\frac{Y}{Z+X}(C+A)+\frac{Z}{X+Y}(A+B)$$

$$\left(X=y^2z, Y=z^2x, Z=x^2y, A=\frac{1}{x}, B=\frac{1}{y}, C=\frac{1}{z}\right)$$

$$=\frac{X}{Y+Z}.\sqrt{B+C}^2+\frac{Y}{Z+X}.\sqrt{C+A}^2+\frac{Z}{X+Y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\frac{1}{xy}} \overset{\text{Bergstrom}}{\geq} \sqrt{\frac{27}{\sum_{cyc}xy}}$$

$$\geq \sqrt{\frac{27}{\sum_{cyc}x^2}}=\sqrt{\frac{27}{\sum_{cyc}(b+c)}}=\frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}$$

$$\therefore \sqrt{\frac{b+c}{a+b}}.\frac{\sqrt{b+c}+\sqrt{a+c}}{\sqrt{(a+b)(b+c)}+(a+c)}+\sqrt{\frac{a+c}{b+c}}.\frac{\sqrt{a+c}+\sqrt{a+b}}{\sqrt{(b+c)(a+c)}+(a+b)}$$

$$+\sqrt{\frac{a+b}{a+c}}.\frac{\sqrt{a+b}+\sqrt{b+c}}{\sqrt{(a+c)(a+b)}+(b+c)} \geq \frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}$$

$$\forall\, a,b,c>0, '' ='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Tapas Das-India

Let $a+b=x, b+c=y, c+a=z,$ *then the given problem written as*

$$\sqrt{\frac{y}{x}}.\frac{\sqrt{y}+\sqrt{z}}{\sqrt{xy}+z}+\sqrt{\frac{z}{y}}.\frac{\sqrt{x}+\sqrt{z}}{\sqrt{yz}+x}+\sqrt{\frac{x}{z}}.\frac{\sqrt{y}+\sqrt{x}}{\sqrt{xz}+y} \geq \frac{3\sqrt{3}}{\sqrt{x+y+z}}$$

Now $\sqrt{\frac{y}{x}}.\frac{\sqrt{y}+\sqrt{z}}{\sqrt{xy}+z}=\frac{y+\sqrt{yz}}{\sqrt{x}(z+\sqrt{xy})}, \sqrt{\frac{z}{y}}.\frac{\sqrt{x}+\sqrt{z}}{\sqrt{yz}+x}=\frac{z+\sqrt{xz}}{\sqrt{y}(x+\sqrt{yz})}, \sqrt{\frac{x}{z}}.\frac{\sqrt{y}+\sqrt{x}}{\sqrt{xz}+y}$

$$=\frac{x+\sqrt{yx}}{\sqrt{z}(\sqrt{xz}+y)}, \textit{Now} \sum\sqrt{\frac{y}{x}}.\frac{\sqrt{y}+\sqrt{z}}{\sqrt{xy}+z}$$

$$=\sum\frac{y+\sqrt{yz}}{\sqrt{x}(z+\sqrt{xy})} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod\frac{y+\sqrt{yz}}{\sqrt{x}(z+\sqrt{xy})}}$$

$$=3\sqrt[3]{\frac{1}{\sqrt{xyz}}} \overset{AM-GM}{\geq} \frac{3}{\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}} \overset{CBS}{\geq} \frac{9}{\sqrt{3(x+y+z)}}$$

$$=\frac{3\sqrt{3}}{\sqrt{x+y+z}}$$ *Equality holds for* $x=y=z$ *or* $a=b=c$